Probabilistic Graphical Models in Computer Vision

Lecture: Dr. Tao Wu
Exercises: Yuesong Shen, Zhenzhang Ye
Summer Semester 2019

Computer Vision Group
Institut für Informatik
Technische Universität München

Weekly Exercises 7

Room: 02.09.023

Wednesday, 03.07.2019, 12:15 - 14:00

MCMC sampling (Due: 01.07) (14+4 Points)

Exercise 1 (4 Points)**.** (regular $\Rightarrow$ irreducible and aperiodic) Show that if a Markov chain is regular, then it is irreducible and aperiodic.

Solution. Let $\pi(x|x')$ be the transition kernel of a regular Markov chain, then $\exists n$ such that $p(x_n|x_0) > 0, \forall x_0, x_n$.

- Irreducibility: $\forall x, x' \in \mathcal{X}$ we know that $p(X_n = x'|X_0 = x) > 0$, thus the Markov chain is irreducible.
- Aperiodicity: $\forall x, x' \in \mathcal{X}$, we know that $p(X_n = x'|X_0 = x) > 0$, thus we have $p(X_n = x|X_0 = x) > 0$. Also $p(X_{n+1} = x|X_0 = x) = \sum_{x'} p(X_{n+1} = x|X_n = x')p(X_n = x'|X_0 = x)$. There must be one x' such that $p(X_{n+1} = x|X_n = x') > 0$, otherwise $\forall x', \pi(x|x') = 0$ and the Markov chain is clearly not irreducible since the state x can never be reached. Thus $p(X_{n+1} = x|X_0 = x) = \sum_{x'} p(X_{n+1} = x|X_n = x')p(X_n = x'|X_0 = x) > 0$. Since $n, n+1$ are consecutive, their GCD is 1, and thus the Markov chain is aperiodic.

Exercise 2 (8 Points)**.** (Properties of transition matrix) For Markov chains with finite state spaces, the transition kernel can be represented in matrix form: $T_{l,m} = Pr(X_i = m|X_{i-1} = l)$ for any time step i. For each of the transition matrices in the following, determine whether it is irreducible / aperiodic:

$$\text{(a)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{(d)} \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{bmatrix}$$

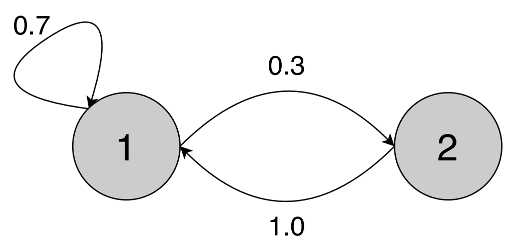
Solution. (a). Not irreducible because transition between any different states has zero probability, it is aperiodic (always the same state);

(b). Not irreducible because transition between state 1 and state 2 or 3 has zero probability, it is also not aperiodic since there is a rotation between state 2 and 3 for every 2 steps;

(c). Irreducible (all transition has probability 1 for either n=1, 2 or 3), however not aperiodic since there is a rotation between the 3 states every 3 steps;

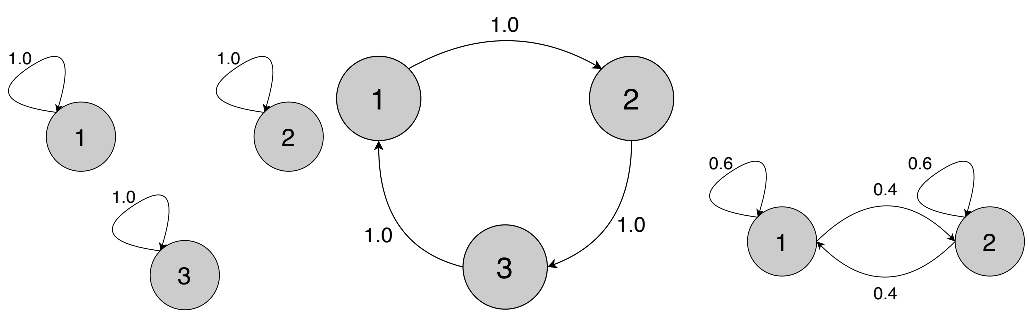
(d). Irreducible and aperiodic, since it is regular;

Exercise 3 (6 Points)**.** (State transition diagram and stationary distribution) A transition matrix of a finite state Markov chain can be visualized as a diagram with nodes representing the states and directed arcs the possibles transitions and its probability. For example, a diagram for the transition matrix $\begin{bmatrix} 0.7 & 0.3 \\ 1.0 & 0.0 \end{bmatrix}$ looks like the following:



For each of the transition matrices (a), (c), (d) in Exercise 2, draw a corresponding diagram, write down the set of stationary distributions, and for each stationary distribution indicate the set of initial distributions that will converge towards it.

Solution. Example diagrams are shown in the following:



(a) (c) (d)

(a). In this case, the Markov chain stays constant and every possible distribution is a stationary distribution with itself the only initial distribution that converges towards it.

(c). In this case, the Markov chain is rotating the coordinates of the distribution, thus the only stationary distribution is $(1/3, 1/3, 1/3)$ which is attainable only via the initial distribution $(1/3, 1/3, 1/3)$.

(d). In this case we have an irreducible and aperiodic Markov chain, thus all initial state will converge to a unique stationary distribution, which in this case is $(0.5, 0.5)$.

Programming (Due:08.07) (12 Points)

In this programming exercise, you are asked to use mean field and Gibbs sampling for image segmentation. See the ipython file for more details.