

Weekly Exercises 5

Room: 02.09.023

Wednesday, 19.06.2019, 12:15 - 14:00

Junction Tree Algorithm (Due:17.06) (12+6 Points)

Exercise 1 (4 Points). Given a pairwise MRF $\mathcal{H} = (\mathcal{V}, \mathcal{E})$,

1. show that the messages between two variable nodes in Belief Propagation can be simplified as:

$$m_{i \rightarrow j}(y_j) = \log \sum_{y_i} \exp \left(-E_i(y_i) - E_{ij}(y_i, y_j) + \sum_{k \in \mathcal{N}_{\mathcal{H}}(i) \setminus \{j\}} m_{k \rightarrow i}(y_i) \right). \quad (1)$$

Hint: Firstly, consider a leaf node in the tree. Then assume the message on a node satisfies the equation, show that the message to its parent satisfies as well.

2. show that the node marginal can be computed by:

$$\mu_i(y_i) = \exp \left(-E_i(y_i) + \sum_{k \in \mathcal{N}_{\mathcal{H}}(i)} m_{k \rightarrow i}(y_i) - \log Z \right). \quad (2)$$

Solution. 1. The idea of our proof is to use the message computation in lecture slides, show it will equal to Eq.(1).

Assume variable node j is a node which has only one unary factor node as descendant. Applying the message equation in lecture slides, the message from unary factor F_u is then:

$$r_{F_u \rightarrow j}(y_j) = \log \exp(-E_j(y_j)).$$

If we use the Eq.(1) where i is empty, we can get the same result.

Now consider a node j which is an intermediate node. Assume all its descending variable nodes has message $m_{k \rightarrow i}(y_i)$. The message from all descending pairwise factor nodes F_p to variable node j is then:

$$r_{F_p \rightarrow j}(y_j) = \log \sum_{y_i} \exp \left(-E_i(y_i) - E_{i,j}(y_i, y_j) + \sum_{k \in \mathcal{N}_{\mathcal{H}}(i)} m_{k \rightarrow i}(y_i) \right).$$

where $E_i(y_i)$ comes from the unary factor and $E_{i,j}(y_i, y_j)$ from pairwise factor.

2. We know that the variable marginals are:

$$\begin{aligned}
 p(y_i) &= \exp \left(\sum_{F \in \mathcal{N}_{\mathcal{H}}(i)} r_{F \rightarrow i}(y_i) - \log Z \right) \\
 &= \exp \left(r_{F_u \rightarrow i}(y_i) + \sum_{F_p \in \mathcal{N}_{\mathcal{H}}(i)} r_{F_p \rightarrow i}(y_i) - \log Z \right) \\
 &= \exp \left(-E_i(y_i) + \sum_{k \in \mathcal{N}_{\mathcal{H}}(i)} m_{k \rightarrow i}(y_i) - \log Z \right)
 \end{aligned}$$

Exercise 2 (4 Points). Considering the same graph in sheet 4 as shown in Figure 1 and assuming we know $\phi(X_1, X_2, X_3)$, here we use Belief Propagation to compute $p(x_5)$:

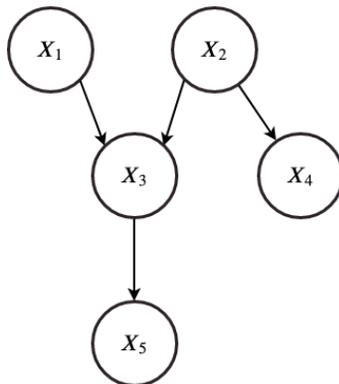


Figure 1: Exercise 2

1. Draw the corresponding factor graph. Try to keep the tree structure in factor graph.
2. Apply Belief Propagation to compute $p(x_5)$. Write down the messages passed in whole graph and the final formula to compute $p(x_5)$.

Solution. 1. The corresponding factor graph is in Figure 2.

2. Since the factor graph is already a tree, we can apply Belief Propagation directly.
Pick X_5 as the root (because we only need one direction passing in this case)

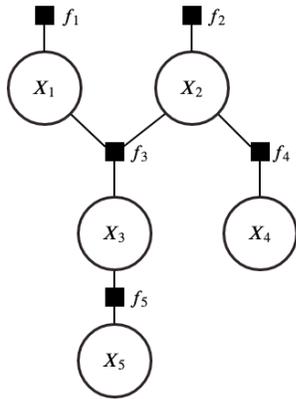


Figure 2: Solution 2.1

and X_4, X_1 will be the leaf nodes, we compute the messages:

$$\begin{aligned}
 q_{X_4 \rightarrow F_4}(X_4) &= 0 \\
 r_{F_4 \rightarrow X_2}(X_2) &= \log \sum_{X_4} \exp(-E_{F_4}(X_4)) \\
 r_{F_1 \rightarrow X_1}(X_1) &= -E_{F_1}(X_1) \\
 q_{X_1 \rightarrow F_3}(X_1) &= r_{F_1 \rightarrow X_1}(X_1) \\
 r_{F_2 \rightarrow X_2}(X_2) &= -E_{F_2}(X_2) \\
 q_{X_2 \rightarrow F_3}(X_2) &= r_{F_4 \rightarrow X_2} + r_{F_2 \rightarrow X_2} \\
 r_{F_3 \rightarrow X_3}(X_3) &= \log \sum_{X_1, X_2} \exp(-E_{F_3}(X_1, X_2) + \sum_{i=1,2} q_{X_i \rightarrow F_3}(X_i)) \\
 q_{X_3 \rightarrow F_5}(X_3) &= r_{F_5 \rightarrow X_3} \\
 r_{F_5 \rightarrow X_5}(X_5) &= \log \sum_{X_3} \exp(-E_{F_5}(X_3) + q_{X_3 \rightarrow F_5}(X_3))
 \end{aligned}$$

The log partition function is:

$$\log Z = \log \sum_{X_5} \exp(r_{F_5 \rightarrow X_5}(X_5))$$

Therefore, we can compute $p(x_5)$:

$$p(x_5) = \exp(r_{F_5 \rightarrow X_5}(x_5) - \log Z)$$

If we want to get the marginals of other variables, we need to perform message passing from root to leaves. Then use the same formula to compute them.

Exercise 3 (6 Points). Consider the bayesian network in Figure 3:

1. Moralize and triangulate it. Draw the graph.
2. Draw the junction tree with order x_1, x_2, x_8, x_3, x_7 .

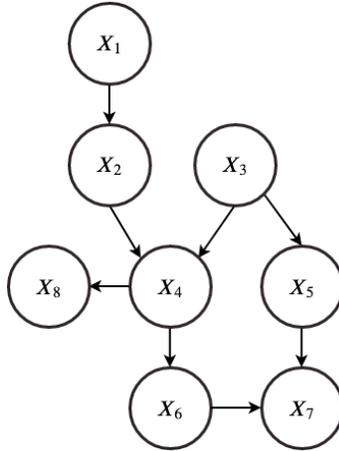
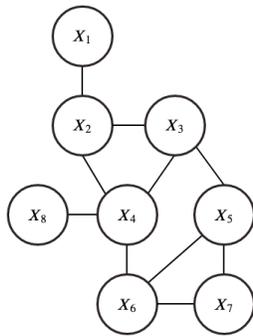
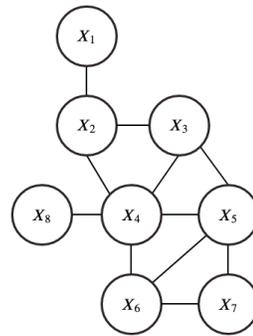


Figure 3: Exercise 3



(a) moralized graph



(b) triangulated graph

Figure 4: Solution 3.1

Solution. 1. As shown in Figure 4, left is moralized graph and right is after triangulation.

2. The corresponding junction tree with that order is in Figure 5

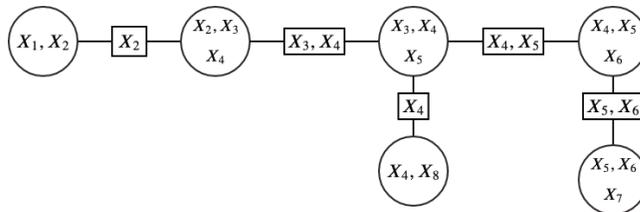


Figure 5: Solution 3.2

Exercise 4 (4 Points). Given two distributions q, p , show the following properties of KL:

$$\text{KL}(q|p) \geq 0, \forall q, p$$

Hint: First show that $\log x \leq x - 1$ and you can assume we use the natural logarithm here.

Solution. Let $f(x) = x - 1 - \log x$. Then $f'(x) = 1 - \frac{1}{x}$ which is negative when $0 < x < 1$ and positive when $x > 1$. Therefore, the minimal value of $f(x)$ is $f(1) = 0$, which states that $\log x \leq x - 1$.

$$\begin{aligned}\text{KL}(q|p) &= \sum_y q(y) \log \frac{q(y)}{p(y)} \\ &= - \sum_y q(y) \log \frac{p(y)}{q(y)} \\ &\geq - \sum_y q(y) \left(\frac{p(y)}{q(y)} - 1 \right) \\ &= - \sum_y (p(y) - 1) \\ &\geq 0\end{aligned}$$

Programming(Due:24.06)

(12 Points)

In this programming exercise, you are asked to implement belief propagation. See the ipython file for more details.