



# Multiple View Geometry: Exercise Sheet 10

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## Part I: Theory

### 1. Variational Calculus and Euler-Lagrange

(a) Under the assumption that  $h$  vanishes at the boundary of  $\Omega$ , prove that

$$\frac{dE(u)}{du} = \frac{\partial \mathcal{L}(u, \nabla u)}{\partial u} - \operatorname{div} \left( \frac{\partial \mathcal{L}(u, \nabla u)}{\partial (\nabla u)} \right).$$

We can expand  $\mathcal{L}(u + \epsilon h, \nabla u + \epsilon \nabla h)$  in terms of  $\epsilon$ :

$$\mathcal{L}(u + \epsilon h, \nabla u + \epsilon \nabla h) = \mathcal{L}(u, \nabla u) + \epsilon h \frac{\partial \mathcal{L}}{\partial u} + \epsilon \nabla h \frac{\partial \mathcal{L}}{\partial (\nabla u)} + \mathcal{O}(\epsilon^2)$$

Inserting into  $\left. \frac{\delta E(u)}{\delta u} \right|_h$  gives

$$\left. \frac{\delta E(u)}{\delta u} \right|_h = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{\Omega} \left( \epsilon h(x) \left. \frac{\partial \mathcal{L}}{\partial u} \right|_{u(x)} + \epsilon \nabla h(x) \left. \frac{\partial \mathcal{L}}{\partial (\nabla u)} \right|_{\nabla u(x)} + \mathcal{O}(\epsilon^2) \right) dx$$

The  $\epsilon$  in the first two terms cancels, and in the  $\mathcal{O}(\epsilon^2)$  it will go to zero for  $\epsilon \rightarrow 0$ . Integration by parts of the second term yields

$$\begin{aligned} \int_{\Omega} \nabla h(x) \frac{\partial \mathcal{L}}{\partial (\nabla u)} dx &= \int_{\partial \Omega} h(x) \frac{\partial \mathcal{L}}{\partial (\nabla u)} ds - \int_{\Omega} h(x) \operatorname{div} \left( \frac{\partial \mathcal{L}}{\partial (\nabla u)} \right) dx = \\ &= - \int_{\Omega} h(x) \operatorname{div} \left( \frac{\partial \mathcal{L}}{\partial (\nabla u)} \right) dx. \end{aligned}$$

Thus,

$$\left. \frac{\delta E(u)}{\delta u} \right|_h = \int_{\Omega} h(x) \left( \frac{\partial \mathcal{L}}{\partial u} - \operatorname{div} \left( \frac{\partial \mathcal{L}}{\partial (\nabla u)} \right) \right) dx \Rightarrow \text{claim}.$$

(b) Which condition must hold true for a minimizer  $u_0$  of  $E(u)$  ...

- ... in general?

$$\frac{dE(u)}{du} = 0 \Rightarrow \frac{\partial \mathcal{L}(u, \nabla u)}{\partial u} = \operatorname{div} \left( \frac{\partial \mathcal{L}(u, \nabla u)}{\partial (\nabla u)} \right).$$

- ... if  $\mathcal{L}(u, \nabla u) = \mathcal{L}(u)$ ?

$$\frac{\partial \mathcal{L}(u)}{\partial u} = 0.$$

- ... if  $\mathcal{L}(u, \nabla u) = \mathcal{L}(\nabla u)$ ?

$$\operatorname{div} \left( \frac{\partial \mathcal{L}(\nabla u)}{\partial(\nabla u)} \right) = 0.$$

## 2. Multiview Reconstruction as Shape Optimization

(a) Write down the Euler-Lagrange equation for the given energy  $E(u)$ .

The E-L equations are

$$\frac{dE(u)}{du} = -\operatorname{div} \left( \frac{\partial \mathcal{L}(\nabla u)}{\partial(\nabla u)} \right) = 0 \quad \text{with} \quad \mathcal{L}(\nabla u) = \rho |\nabla u|$$

Taking the derivative w.r.t.  $\nabla u$  gives

$$0 = -\operatorname{div} \left( \rho(x) \frac{\nabla u(x)}{|\nabla u(x)|} \right).$$

It is also possible (but not necessarily required) to expand this further using the product rule for divergence:

$$\begin{aligned} \operatorname{div} \left( \rho(x) \frac{\nabla u(x)}{|\nabla u(x)|} \right) &= \nabla \rho(x)^\top \frac{\nabla u(x)}{|\nabla u(x)|} + \rho(x) \operatorname{div} \left( \frac{\nabla u(x)}{|\nabla u(x)|} \right) \\ &= \nabla \rho(x)^\top \frac{\nabla u(x)}{|\nabla u(x)|} + \rho(x) \frac{\nabla^2 u(x) - 1}{|\nabla u(x)|} \end{aligned}$$

(b) Write down one gradient descent iteration for  $E(u)$ .

$$u^{(k+1)}(x) = u^{(k)}(x) - \tau \frac{dE(u)}{du} = u^{(k)}(x) + \tau \left( \nabla \rho(x)^\top \frac{\nabla u(x)}{|\nabla u(x)|} + \rho(x) \frac{\nabla^2 u(x) - 1}{|\nabla u(x)|} \right)$$