



Multiple View Geometry: Solution Sheet 3

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Part I: Theory

$$1. \quad (a) \quad M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(b) \quad M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(c) \quad M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(d) \quad M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & RT \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_1 t_x \\ r_{21} & r_{22} & r_{23} & r_2 t_y \\ r_{31} & r_{32} & r_{33} & r_3 t_z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where r_1, r_2, r_3 are the row vectors of R : $R = \begin{pmatrix} -r_1 - \\ -r_2 - \\ -r_3 - \end{pmatrix}$.

$$2. \quad \text{Let } M := (M_1 - M_2) =: \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}.$$

" \Rightarrow ":

We show that \bar{M} is skew-symmetric by distinguishing diagonal and off-diagonal elements of M :

$$(a) \quad \forall i: 0 = e_i^\top M e_i = m_{ii}$$

where $e_i = i$ -th unit vector

$$(b) \quad \forall i \neq j: 0 = (e_i + e_j)^\top M (e_i + e_j)$$

where $e_j = j$ -th unit vector

$$= m_{ii} + m_{jj} + m_{ij} + m_{ji} \Rightarrow m_{ij} = -m_{ji}$$

hence, $m_{ii} = 0$ and $m_{ij} = -m_{ji}$, i.e. M is skew-symmetric.

” \Leftarrow ”:

using $M = -M^\top$, we directly calculate

$$\begin{aligned}\forall x: x^\top Mx &= (x^\top Mx)^\top = x^\top M^\top x = -(x^\top Mx) \\ &\Rightarrow x^\top Mx = 0\end{aligned}$$

Alternative for ” \Leftarrow ”:

$$\forall x: x^\top Mx = x^\top (\check{M} \times x) = 0$$

Because M is skew-symmetric, Mx can be interpreted as a cross product. The result of any cross product with x is orthogonal to x .

3. We know: $\omega = (\omega_1 \ \omega_2 \ \omega_3)^\top$ with $\|\omega\| = 1$ and $\hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$.

(a)

$$\begin{aligned}\hat{\omega}^2 &= \begin{pmatrix} -(\omega_2^2 + \omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2 + \omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2 + \omega_2^2) \end{pmatrix} \\ &= \begin{pmatrix} \omega_1^2 - \underbrace{(\omega_2^2 + \omega_3^2)}_1 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 - \underbrace{(\omega_1^2 + \omega_3^2)}_1 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 - \underbrace{(\omega_1^2 + \omega_2^2)}_1 \end{pmatrix} \\ &= \begin{pmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \omega\omega^\top - \mathbf{I}\end{aligned}$$

$$\begin{aligned}\hat{\omega}^3 &= \hat{\omega} \hat{\omega}^2 \\ &= \hat{\omega} (\omega\omega^\top - \mathbf{I}) \\ &= \hat{\omega} \omega (\omega^\top) - \hat{\omega} \mathbf{I} \\ &= (\omega \times \omega) \omega^\top - \hat{\omega} \\ &= -\hat{\omega} \quad (\text{as } \omega \times \omega = 0)\end{aligned}$$

Alternative solution for $\hat{\omega}^3$:

$$\begin{aligned}
 \hat{\omega}^3 &= \begin{pmatrix} -(\omega_2^2 + \omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2 + \omega_2^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2 + \omega_2^2) \end{pmatrix} \cdot \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & \omega_3 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & -\omega_2 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 \\ -\omega_3 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & 0 & \omega_1 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 \\ \omega_2 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & -\omega_1 \cdot \underbrace{(\omega_1^2 + \omega_2^2 + \omega_3^2)}_1 & 0 \end{pmatrix} \\
 &= -\hat{\omega}
 \end{aligned}$$

(b) The formulas for n even and odd can be found by writing down the solutions for $n = 1, \dots, 6$:

$$\begin{aligned}
 \hat{\omega} & \\
 \hat{\omega}^2 & \\
 \hat{\omega}^3 &= -\hat{\omega} \\
 \hat{\omega}^4 &= -\hat{\omega}^2 & \text{as: } \hat{\omega}^4 &= \hat{\omega}^3\hat{\omega} = -\hat{\omega}\hat{\omega} = -\hat{\omega}^2 \\
 \hat{\omega}^5 &= \hat{\omega} & \text{as: } \hat{\omega}^5 &= \hat{\omega}^4\hat{\omega} = -\hat{\omega}^2\hat{\omega} = -\hat{\omega}^3 = -(-\hat{\omega}) = \hat{\omega} \\
 \hat{\omega}^6 &= \hat{\omega}^2 & \text{as: } \hat{\omega}^6 &= \hat{\omega}^5\hat{\omega} = \hat{\omega}\hat{\omega} = \hat{\omega}^2
 \end{aligned}$$

For even numbers:

$$\begin{aligned}
 \hat{\omega}^2 & \\
 \hat{\omega}^4 &= -\hat{\omega}^2 \\
 \hat{\omega}^6 &= \hat{\omega}^2
 \end{aligned}$$

For odd numbers:

$$\begin{aligned}
 \hat{\omega} & \\
 \hat{\omega}^3 &= -\hat{\omega} \\
 \hat{\omega}^5 &= \hat{\omega}
 \end{aligned}$$

$$\begin{aligned}
 \text{even: } \hat{\omega}^{2n} &= (-1)^{n+1} \hat{\omega}^2 \quad \text{for } n \geq 1 \\
 \text{odd: } \hat{\omega}^{2n+1} &= (-1)^n \hat{\omega} \quad \text{for } n \geq 0
 \end{aligned}$$

Proof via complete induction:

i. For even numbers $2n$ where $n \geq 1$:

- $n = 1$: $\hat{\omega}^2 = (-1)^2 \hat{\omega}^2$
- Induction step $n \rightarrow n + 1$:

$$\begin{aligned}
 \hat{\omega}^{2(n+1)} &= \hat{\omega}^{2n} \cdot \hat{\omega}^2 \\
 &= (-1)^{n+1} \cdot \hat{\omega}^2 \cdot \hat{\omega}^2 & \text{(assumption)} \\
 &= (-1)^{n+1} \cdot \hat{\omega}^3 \cdot \hat{\omega} \\
 &\stackrel{(a)}{=} (-1)^{(n+1)+1} \cdot \hat{\omega}^2
 \end{aligned}$$

ii. For odd numbers $2n + 1$ where $n \geq 0$:

- $n = 0$: $\hat{\omega}^1 = (-1)^0 \hat{\omega}$
- Induction step $n \rightarrow n + 1$:

$$\begin{aligned}
 \hat{\omega}^{2(n+1)+1} &= \hat{\omega}^{2n+1} \cdot \hat{\omega}^2 \\
 &= (-1)^n \cdot \hat{\omega} \cdot \hat{\omega}^2 && \text{(assumption)} \\
 &= (-1)^n \cdot \hat{\omega}^3 \\
 &\stackrel{(a)}{=} (-1)^{n+1} \cdot \hat{\omega}
 \end{aligned}$$

(c) We know: $\omega \in \mathbb{R}^3$. Let $\nu = \frac{\omega}{\|\omega\|}$ and $t = \|\omega\|$. Hence, $w = \nu t$, $\hat{\omega} = \hat{\nu} t$.

$$\begin{aligned}
 e^{\hat{\omega}} &= e^{\hat{\nu} t} \\
 &= \sum_{n=0}^{\infty} \frac{(\hat{\nu} t)^n}{n!} \\
 &= I + \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \hat{\nu}^{2n} + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \hat{\nu}^{2n+1} \\
 &\stackrel{(b)}{=} I + \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!} \hat{\nu}^2}_{1 - \cos(t)} + \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \hat{\nu}}_{\sin(t)} \\
 &\stackrel{(\text{def.})}{=} I + \frac{\hat{\omega}^2}{\|\omega\|^2} (1 - \cos(\|\omega\|)) + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|)
 \end{aligned}$$