Chapter 2
Representing a Moving Scene

Multiple View Geometry
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The Origins of 3D Reconstruction

The goal to reconstruct the three-dimensional structure of the world from a set of two-dimensional views has a long history in computer vision. It is a classical ill-posed problem, because the reconstruction consistent with a given set of observations or images is typically not unique. Therefore, one will need to impose additional assumptions.
Mathematically, the study of geometric relations between a 3D scene and the observed 2D projections is based on two types of transformations, namely:

- Euclidean motion or rigid-body motion representing the motion of the camera from one frame to the next.
- Perspective projection to account for the image formation process (see pinhole camera, etc).

The notion of perspective projection has its roots among the ancient Greeks (Euclid of Alexandria, $\sim$ 400 B.C.) and the Renaissance period (Brunelleschi & Alberti, 1435). The study of perspective projection lead to the field of projective geometry (Girard Desargues 1648, Gaspard Monge 18th cent.).

The Origins of 3D Reconstruction

The first work on the problem of multiple view geometry was that of Erwin Kruppa (1913) who showed that two views of five points are sufficient to determine both the relative transformation (motion) between the two views and the 3D location (structure) of the points up to finitely many solutions.

A linear algorithm to recover structure and motion from two views based on the epipolar constraint was proposed by Longuet-Higgins in 1981. An entire series of works along these lines was summarized in several text books (Faugeras 1993, Kanatani 1993, Maybank 1993, Weng et al. 1993).

Extensions to three views were developed by Spetsakis and Aloimonos '87, '90, and by Shashua '94 and Hartley '95. Factorization techniques for multiple views and orthogonal projection were developed by Tomasi and Kanade 1992.

The joint estimation of camera motion and 3D location is called structure and motion or (more recently) visual SLAM.

Three-dimensional Euclidean Space

The three-dimensional Euclidean space $\mathbb{E}^3$ consists of all points $p \in \mathbb{E}^3$ characterized by coordinates

$$\boldsymbol{X} \equiv (X_1, X_2, X_3)^\top \in \mathbb{R}^3,$$

such that $\mathbb{E}^3$ can be identified with $\mathbb{R}^3$. That means we talk about points ($\mathbb{E}^3$) and coordinates ($\mathbb{R}^3$) as if they were the same thing. Given two points $\boldsymbol{X}$ and $\boldsymbol{Y}$, one can define a bound vector as

$$v = \boldsymbol{X} - \boldsymbol{Y} \in \mathbb{R}^3.$$

Considering this vector independent of its base point $\boldsymbol{Y}$ makes it a free vector. The set of free vectors $v \in \mathbb{R}^3$ forms a linear vector space. By identifying $\mathbb{E}^3$ and $\mathbb{R}^3$, one can endow $\mathbb{E}^3$ with a scalar product, a norm and a metric. This allows to compute distances, curve length

$$l(\gamma) \equiv \int_0^1 |\dot{\gamma}(s)| \, ds \quad \text{for a curve } \gamma : [0, 1] \to \mathbb{R}^3,$$

areas or volumes.

Cross Product & Skew-symmetric Matrices

On $\mathbb{R}^3$ one can define a cross product

$$\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 : \quad u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \in \mathbb{R}^3,$$

which is a vector orthogonal to u and v. Since $u \times v = -v \times u$, the cross product introduces an orientation. Fixing u induces a linear mapping $v \mapsto u \times v$ which can be represented by the skew-symmetric matrix

$$\widehat{u} = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

In turn, every skew symmetric matrix $M = -M^\top \in \mathbb{R}^{3\times 3}$ can be identified with a vector $u \in \mathbb{R}^3$. The operator $\widehat{\ }$ defines an isomorphism between $\mathbb{R}^3$ and the space $so(3)$ of all 3×3 skew-symmetric matrices. Its inverse is denoted by $\vee : so(3) \to \mathbb{R}^3$.

Rigid-body Motion

A rigid-body motion (or rigid-body transformation) is a family of maps

$$g_t : \mathbb{R}^3 \to \mathbb{R}^3; \quad \boldsymbol{X} \mapsto g_t(\boldsymbol{X}), \quad t \in [0, T]$$

which preserve the norm and cross product of any two vectors:

- $|g_t(v)| = |v|, \quad \forall\, v \in \mathbb{R}^3,$
- $g_t(u) \times g_t(v) = g_t(u \times v), \quad \forall\, u, v \in \mathbb{R}^3.$

Since norm and scalar product are related by the polarization identity

$$\langle u, v \rangle = \frac{1}{4}(|u + v|^2 - |u - v|^2),$$

one can also state that a rigid-body motion is a map which preserves inner product and cross product. As a consequence, rigid-body motions also preserve the triple product

$$\langle g_t(u), g_t(v) \times g_t(w) \rangle = \langle u, v \times w \rangle, \quad \forall\, u, v, w \in \mathbb{R}^3,$$

which means that they are volume-preserving.

Representation of Rigid-body Motion

Does the above definition lead to a mathematical representation of rigid-body motion?

Since it preserves lengths and orientation, the motion g_t of a rigid body is sufficiently defined by specifying the motion of a Cartesian coordinate frame attached to the object (given by an origin and orthonormal oriented vectors $e_1, e_2, e_3 \in \mathbb{R}^3$). The motion of the origin can be represented by a translation $T \in \mathbb{R}^3$, whereas the transformation of the vectors e_i is given by new vectors $r_i = g_t(e_i)$.

Scalar and cross product of these vectors are preserved:

$$r_i^\top r_j = g_t(e_i)^\top g_t(e_j) = e_i^\top e_j = \delta_{ij}, \quad r_1 \times r_2 = r_3.$$

The first constraint amounts to the statement that the matrix $R = (r_1, r_2, r_3)$ is an orthogonal (rotation) matrix: $R^\top R = RR^\top = I$, whereas the second property implies that $\det(R) = +1$, in other words: R is an element of the group $SO(3) = \left\{R \in \mathbb{R}^{3\times 3} \mid R^\top R = I,\ \det(R) = +1\right\}$.
Thus the rigid body motion g_t can be written as:

$$g_t(x) = Rx + T.$$

Exponential Coordinates of Rotation

We will now derive a representation of an infinitesimal rotation. To this end, consider a family of rotation matrices $R(t)$ which continuously transform a point from its original location ($R(0) = I$) to a different one.

$$\boldsymbol{X}_{trans}(t) = R(t)\boldsymbol{X}_{orig}, \quad \text{with } R(t) \in SO(3).$$

Since $R(t)R(t)^\top = I$, $\forall t$, we have

$$\frac{d}{dt}(RR^\top) = \dot{R}R^\top + R\dot{R}^\top = 0 \quad \Rightarrow \quad \dot{R}R^\top = -(\dot{R}R^\top)^\top.$$

Thus, $\dot{R}R^\top$ is a skew-symmetric matrix. As shown in the section about the $\widehat{\cdot}$-operator, this implies that there exists a vector $w(t) \in \mathbb{R}^3$ such that:

$$\dot{R}(t)R^\top(t) = \widehat{w}(t) \quad \Leftrightarrow \quad \dot{R}(t) = \widehat{w}R(t).$$

Since $R(0) = I$, it follows that $\dot{R}(0) = \widehat{w}(0)$. Therefore the skew-symmetric matrix $\widehat{w}(0) \in so(3)$ gives the first order approximation of a rotation:

$$R(dt) = R(0) + dR = I + \widehat{w}(0)\,dt.$$

Lie Group and Lie Algebra

The above calculations showed that the effect of any infinitesimal rotation $R \in SO(3)$ can be approximated by an element from the space of skew-symmetric matrices

$$so(3) = \{\widehat{w} \mid w \in \mathbb{R}^3\}.$$

The rotation group $SO(3)$ is called a Lie group. The space $so(3)$ is called its Lie algebra.

Def.: A Lie group (or infinitesimal group) is a smooth manifold that is also a group, such that the group operations multiplication and inversion are smooth maps.

As shown above: The Lie algebra $so(3)$ is the tangent space at the identity of the rotation group $SO(3)$.

An algebra over a field K is a vector space V over K with multiplication on the space V. Elements $\widehat{w}$ and $\widehat{v}$ of the Lie algebra generally do not commute.
One can define the Lie bracket

$$[\,.\,,\,.\,] : so(3) \times so(3) \to so(3); \quad [\widehat{w}, \widehat{v}] \equiv \widehat{w}\widehat{v} - \widehat{v}\widehat{w}.$$

Sophus Lie (1841 - 1899)

Marius Sophus Lie was a Norwegian-born mathematician. He created the theory of continuous symmetry, and applied it to the study of geometry and differential equations. Among his greatest achievements was the discovery that continuous transformation groups are better understood in their linearized versions ("Theorie der Transformationsgruppen" 1893). These infinitesimal generators form a structure which is today known as a Lie algebra. The linearized version of the group law corresponds to an operation on the Lie algebra known as the commutator bracket or Lie bracket. 1882 Professor in Christiania (Oslo), 1886 Leipzig (succeeding Felix Klein), 1898 Christiania.

The Exponential Map

Given the infinitesimal formulation of rotation in terms of the skew symmetric matrix $\widehat{w}$, is it possible to determine a useful representation of the rotation $R(t)$? Let us assume that $\widehat{w}$ is constant in time.

The differential equation system

$$\begin{cases} \dot{R}(t) = \widehat{w}R(t), \\ R(0) = I. \end{cases}$$

has the solution

$$R(t) = e^{\widehat{w}t} = \sum_{n=0}^{\infty} \frac{(\widehat{w}t)^n}{n!} = I + \widehat{w}t + \frac{(\widehat{w}t)^2}{2!} + \dots,$$

which is a rotation around the axis $w \in \mathbb{R}^3$ by an angle of t (if $\|w\| = 1$). Alternatively, one can absorb the scalar $t \in \mathbb{R}$ into the skew symmetric matrix $\widehat{w}$ to obtain $R(t) = e^{\widehat{v}}$ with $\widehat{v} = \widehat{w}t$. This matrix exponential therefore defines a map from the Lie algebra to the Lie group:

$$\exp : so(3) \to SO(3); \quad \widehat{w} \mapsto e^{\widehat{w}}.$$

The Logarithm of $SO(3)$

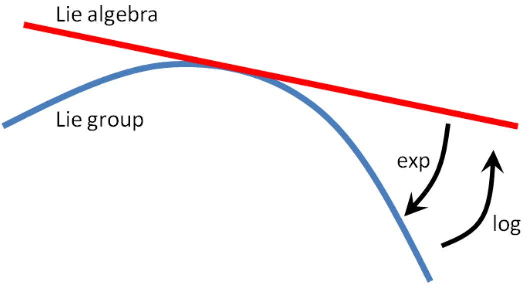
As in the case of real analysis one can define an inverse function to the exponential map by the logarithm. In the context of Lie groups, this will lead to a mapping from the Lie group to the Lie algebra. For any rotation matrix $R \in SO(3)$, there exists a $w \in \mathbb{R}^3$ such that $R = \exp(\widehat{w})$. Such an element is denoted by $\widehat{w} = \log(R)$.
If $R = (r_{ij}) \neq I$, then an appropriate w is given by:

$$|w| = \cos^{-1}\left(\frac{\mathrm{trace}(R) - 1}{2}\right), \quad \frac{w}{|w|} = \frac{1}{2\sin(|w|)}\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}.$$

For $R = I$, we have $|w| = 0$, i.e. a rotation by an angle 0. The above statement says: Any orthogonal transformation $R \in SO(3)$ can be realized by rotating by an angle $|w|$ around an axis $\frac{w}{|w|}$ as defined above. We will not prove this statement.

Obviously the above representation is not unique since increasing the angle by multiples of 2π will give the same rotation R.

Schematic Visualization of Lie Group & Lie Algebra



Definition: A Lie group is a smooth manifold that is also a group, such that the group operations multiplication and inversion are smooth maps.

Definition: The tangent space to a Lie group at the identity element is called the associated Lie algebra.

The mapping from the Lie algebra to the Lie group is called the exponential map. Its inverse is called the logarithm.

Rodrigues' Formula

We have seen that any rotation can be realized by computing $R = e^{\widehat{w}}$. In analogy to the well-known Euler equation

$$e^{i\phi} = \cos(\phi) + i\,\sin(\phi), \quad \forall \phi \in \mathbb{R},$$

we have an expression for skew symmetric matrices $\widehat{w} \in so(3)$:

$$\boxed{e^{\widehat{w}} = I + \frac{\widehat{w}}{|w|}\sin(|w|) + \frac{\widehat{w}^2}{|w|^2}\big(1 - \cos(|w|)\big).}$$

This is known as Rodrigues' formula.

<u>Proof:</u> Let $t = |w|$ and $v = w/|w|$. Then

$$\widehat{v}^2 = vv^\top - I, \quad \widehat{v}^3 = -\widehat{v}, \quad \dots$$

and

$$e^{\widehat{w}} = e^{\widehat{v}t} = I + \underbrace{\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right)}_{\sin(t)}\widehat{v} + \underbrace{\left(\frac{t^2}{2!} - \frac{t^4}{4!} + \frac{t^6}{6!} - \dots\right)}_{1-\cos(t)}\widehat{v}^2.$$

Representation of Rigid-body Motions $SE(3)$

We have seen that the motion of a rigid-body is uniquely determined by specifying the translation T of any given point and a rotation matrix R defining the transformation of an oriented Cartesian coordinate frame at the given point. Thus the space of rigid-body motions given by the group of special Euclidean transformations

$$\boxed{SE(3) \equiv \left\{ g = (R, T) \mid R \in SO(3),\ T \in \mathbb{R}^3 \right\}.}$$

In homogeneous coordinates, we have:

$$\boxed{SE(3) \equiv \left\{ g = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} \middle| R \in SO(3),\ T \in \mathbb{R}^3 \right\} \quad \subset \mathbb{R}^{4\times 4}.}$$

In the context of rigid motions, one can see the difference between points in $\mathbb{E}^3$ (which can be rotated and translated) and vectors in $\mathbb{R}^3$ (which can only be rotated).

The Lie Algebra of Twists

Given a continuous family of rigid-body transformations

$$g : \mathbb{R} \to SE(3); \quad g(t) = \begin{pmatrix} R(t) & T(t) \\ 0 & 1 \end{pmatrix} \quad \in \mathbb{R}^{4\times 4},$$

we consider

$$\dot{g}(t)g^{-1}(t) = \begin{pmatrix} \dot{R}R^\top & \dot{T} - \dot{R}R^\top T \\ 0 & 0 \end{pmatrix} \quad \in \mathbb{R}^{4\times 4}.$$

As in the case of $SO(3)$, the matrix $\dot{R}R^\top$ corresponds to some skew-symmetric matrix $\widehat{w} \in so(3)$.

Defining a vector $v(t) = \dot{T}(t) - \widehat{w}(t)T(t)$, we have:

$$\dot{g}(t)g^{-1}(t) = \begin{pmatrix} \widehat{w}(t) & v(t) \\ 0 & 0 \end{pmatrix} \equiv \widehat{\xi}(t) \in \mathbb{R}^{4\times 4}.$$

The Lie Algebra of Twists

Multiplying with $g(t)$ from the right, we obtain:

$$\dot{g} = \dot{g}g^{-1}g = \widehat{\xi}g.$$

The 4×4-matrix $\widehat{\xi}$ can be viewed as a tangent vector along the curve $g(t)$. $\widehat{\xi}$ is called a twist.

As in the case of $so(3)$, the set of all twists forms a the tangent space which is the Lie algebra

$$\boxed{se(3) \equiv \left\{ \widehat{\xi} = \begin{pmatrix} \widehat{w} & v \\ 0 & 0 \end{pmatrix} \middle| \; \widehat{w} \in so(3), \; v \in \mathbb{R}^3 \right\} \quad \subset \mathbb{R}^{4\times4}.}$$

to the Lie group $SE(3)$.

As before, we can define operators $\wedge$ and $\vee$ to convert between a twist $\widehat{\xi} \in se(3)$ and its twist coordinates $\xi \in \mathbb{R}^6$:

$$\widehat{\xi} \equiv \begin{pmatrix} v \\ w \end{pmatrix}^{\wedge} \equiv \begin{pmatrix} \widehat{w} & v \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{4\times4}, \quad \begin{pmatrix} \widehat{w} & v \\ 0 & 0 \end{pmatrix}^{\vee} = \begin{pmatrix} v \\ w \end{pmatrix} \in \mathbb{R}^6.$$

Exponential Coordinates for $SE(3)$

The twist coordinates $\xi = \binom{v}{w}$ are formed by stacking the linear velocity $v \in \mathbb{R}^3$ (related to translation) and the angular velocity $w \in \mathbb{R}^3$ (related to rotation).
The differential equation system

$$\begin{cases} \dot{g}(t) = \widehat{\xi} g(t), \quad \widehat{\xi} = \text{const.} \\ g(0) = I, \end{cases}$$

has the solution

$$\boxed{g(t) = e^{\widehat{\xi} t} = \sum_{n=0}^{\infty} \frac{(\widehat{\xi} t)^n}{n!}.}$$

For $w = 0$, we have $e^{\widehat{\xi}} = \begin{pmatrix} I & v \\ 0 & 1 \end{pmatrix}$, while for $w \neq 0$ one can show:

$$\boxed{e^{\widehat{\xi}} = \begin{pmatrix} e^{\widehat{w}} & \frac{(I - e^{\widehat{w}})\widehat{w} v + w w^\top v}{|w|^2} \\ 0 & 1 \end{pmatrix}.}$$

Exponential Coordinates for $SE(3)$

The above shows that the exponential map defines a transformation from the Lie algebra $se(3)$ to the Lie group $SE(3)$:

$$\exp: \; se(3) \rightarrow SE(3); \quad \widehat{\xi} \mapsto e^{\widehat{\xi}}.$$

The elements $\widehat{\xi} \in se(3)$ are called the exponential coordinates for $SE(3)$.

Conversely: For every $g \in SE(3)$ there exist twist coordinates $\xi = (v, w) \in \mathbb{R}^6$ such that $g = \exp(\widehat{\xi})$.

Proof: Given $g = (R, T)$, we know that there exists $w \in \mathbb{R}^3$ with $e^{\widehat{w}} = R$. If $|w| \neq 0$, the exponential form of g introduced above shows that we merely need to solve the equation

$$\frac{(I - e^{\widehat{w}})\widehat{w}v + ww^\top v}{|w|^2} = T$$

for the velocity vector $v \in \mathbb{R}^3$. Just as in the case of $SO(3)$, this representation is generally not unique, i.e. there exist many twists $\widehat{\xi} \in se(3)$ which represent the same rigid-body motion $g \in SE(3)$.

Representing the Motion of the Camera

When observing a scene from a moving camera, the coordinates and velocity of points in camera coordinates will change. We will use a rigid-body transformation

$$g(t) = \begin{pmatrix} R(t) & T(t) \\ 0 & 1 \end{pmatrix} \quad \in SE(3)$$

to represent the motion from a fixed world frame to the camera frame at time t. In particular we assume that at time $t = 0$ the camera frame coincides with the world frame, i.e. $g(0) = I$. For any point $\boldsymbol{X}_0$ in world coordinates, its coordinates in the camera frame at time t are:

$$\boxed{\boldsymbol{X}(t) = R(t)\boldsymbol{X}_0 + T(t),}$$

or in the homogeneous representation

$$\boldsymbol{X}(t) = g(t)\boldsymbol{X}_0.$$

Concatenation of Motions over Frames

Given two different times t_1 and t_2, we denote the transformation from the points in frame t_1 to the points in frame t_2 by $g(t_2, t_1)$:

$$\boldsymbol{X}(t_2) = g(t_2, t_1)\boldsymbol{X}(t_1).$$

Obviously we have:

$$\boldsymbol{X}(t_3) = g(t_3, t_2)\boldsymbol{X}_2 = g(t_3, t_2)g(t_2, t_1)\boldsymbol{X}(t_1) = g(t_3, t_1)\boldsymbol{X}(t_1),$$

and thus:

$$g(t_3, t_1) = g(t_3, t_2)g(t_2, t_1).$$

By transferring the coordinates of frame t_1 to coordinates in frame t_2 and back, we see that:

$$\boldsymbol{X}(t_1) = g(t_1, t_2)\boldsymbol{X}(t_2) = g(t_1, t_2)g(t_2, t_1)\boldsymbol{X}(t_1),$$

which must hold for any point coordinates $\boldsymbol{X}(t_1)$, thus:

$$g(t_1, t_2)g(t_2, t_1) = I \quad \Leftrightarrow \quad g^{-1}(t_2, t_1) = g(t_1, t_2).$$

Rules of Velocity Transformation

The coordinates of point $\boldsymbol{X}_0$ in frame t are given by $\boldsymbol{X}(t) = g(t)\boldsymbol{X}_0$. Therefore the velocity is given by

$$\dot{\boldsymbol{X}}(t) = \dot{g}(t)\boldsymbol{X}_0 = \dot{g}(t)g^{-1}(t)\boldsymbol{X}(t)$$

By introducing the twist coordinates

$$\widehat{V}(t) \equiv \dot{g}(t)g^{-1}(t) = \begin{pmatrix} \widehat{w}(t) & v(t) \\ 0 & 0 \end{pmatrix} \quad \in se(3),$$

we get the expression:

$$\boxed{\dot{\boldsymbol{X}}(t) = \widehat{V}(t)\boldsymbol{X}(t).}$$

In simple $3D$-coordinates this gives:

$$\boxed{\dot{\boldsymbol{X}}(t) = \widehat{w}(t)\boldsymbol{X}(t) + v(t).}$$

The symbol $\widehat{V}(t)$ therefore represents the relative velocity of the world frame as viewed from the camera frame.

Transfer Between Frames: The Adjoint Map

Suppose that a viewer in another frame A is displaced relative to the current frame by a transformation g_{xy}: $\boldsymbol{Y} = g_{xy}\boldsymbol{X}(t)$. Then the velocity in this new frame is given by:

$$\dot{\boldsymbol{Y}}(t) = g_{xy}\dot{\boldsymbol{X}}(t) = g_{xy}\widehat{V}(t)\boldsymbol{X}(t) = g_{xy}\widehat{V}g_{xy}^{-1}\boldsymbol{Y}(t).$$

This shows that the relative velocity of points observed from camera frame A is represented by the twist

$$\widehat{V}_y = g_{xy}\,\widehat{V}\,g_{xy}^{-1} \equiv ad_{g_{xy}}(\widehat{V}).$$

where we have introduced the adjoint map on $se(3)$:

$$ad_g : se(3) \to se(3); \quad \widehat{\xi} \mapsto g\,\widehat{\xi}\,g^{-1}.$$

Summary

	Rotation $SO(3)$	Rigid-body $SE(3)$
Matrix repres.	$R \in GL(3):$ $R^\top R = I,$ $\det(R) = 1$	$g = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix}$
3-D coordinates	$\boldsymbol{X} = R\boldsymbol{X}_0$	$\boldsymbol{X} = R\boldsymbol{X}_0 + T$
Inverse	$R^{-1} = R^\top$	$g^{-1} = \begin{pmatrix} R^\top & -R^\top T \\ 0 & 1 \end{pmatrix}$
Exp. repres.	$R = \exp(\widehat{w})$	$g = \exp(\widehat{\xi})$
Velocity	$\dot{\boldsymbol{X}} = \widehat{w}\boldsymbol{X}$	$\dot{\boldsymbol{X}} = \widehat{w}\boldsymbol{X} + v$
Adjoint map	$\widehat{w} \mapsto R\,\widehat{w}\,R^\top$	$\widehat{\xi} \mapsto g\,\widehat{\xi}\,g^{-1}$

Alternative Representations: Euler Angles

In addition to the exponential parameterization, there exist alternative mathematical representations to parameterize rotation matrices $R \in SO(3)$, given by the Euler angles. These are local coordinates, i.e. the parameterization is only correct for a portion of $SO(3)$.

Given a basis $(\widehat{w}_1, \widehat{w}_2, \widehat{w}_3)$ of the Lie algebra $so(3)$, we can define a mapping from $\mathbb{R}^3$ to the Lie group $SO(3)$ by:

$$\alpha : \ (\alpha_1, \alpha_2, \alpha_3) \quad \mapsto \quad \exp(\alpha_1 \widehat{w}_1 + \alpha_2 \widehat{w}_2 + \alpha_3 \widehat{w}_3).$$

The coordinates $(\alpha_1, \alpha_2, \alpha_3)$ are called Lie-Cartan coordinates of the first kind relative to the above basis.

The Lie-Cartan coordinates of the second kind are defined as:

$$\beta : \ (\beta_1, \beta_2, \beta_3) \quad \mapsto \quad \exp(\beta_1 \widehat{w}_1) \ \exp(\beta_2 \widehat{w}_2) \ \exp(\beta_3 \widehat{w}_3).$$

For the basis representing rotation around the z-, y-, x-axis

$$w_1 = (0,0,1)^\top, \quad w_2 = (0,1,0)^\top, \quad w_3 = (1,0,0)^\top,$$

the coordinates $\beta_1, \beta_2, \beta_3$ are called Euler angles.