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Machine Learning for Computer Vision
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Topic: Probabilistic Graphical Models

Exercise 1: Reading a graphical model

We have the following graphical model:

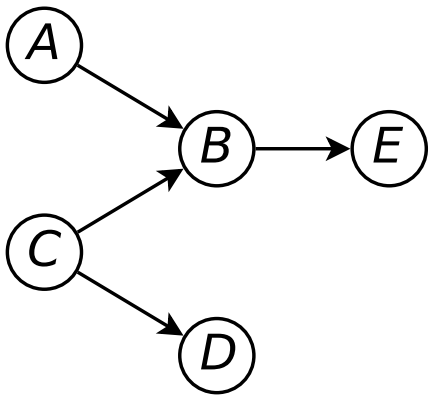


Abbildung 1: Graphical model.

a) Write the joint probability distribution corresponding to the graphical model depicted in Fig. 1.

$$p(A, B, C, D, E) = p(A)p(C)p(B \mid A, C)p(D \mid C)p(E \mid B)$$

b) What are the conditional independence assumptions of this model?

- $A \perp\!\!\!\perp C \mid \emptyset$,
- $D \perp\!\!\!\perp A, B \mid C$,
- $E \perp\!\!\!\perp A, C, D \mid B$.

c) Which of the following assertions are true, and why?

Algorithm to check, whether X is d-seperated from Y by Z (X,Y,Z sets of nodes):

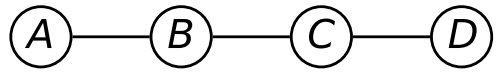
```
boolean is_dsep(X,Y,Z){
foreach x ∈ X, y ∈ Y
 foreach path p connecting x and y
  if (!is_blocked(p,Z)) return false;
 end;
end;
return true;
}

boolean is_blocked(p,Z){
foreach n ∈ p
 if (type(n) == hh)
  if (n ∉ Z ∧ m ∉ Z  ∀  n→ ... →m )
   return true; //case (b)
  end
 else //type(n) == ht or type(n) == tt
  if (n ∈ Z)
   return true; //case (a)
  end
 end
end
return false;
}
```

- B is d-separated from D by C: true (case (a)),
- A is d-separated from C by E: false (case (b) fails as $B \to E$),
- A is d-separated from C by D: true (case (b)),
- E is d-separated from D by B: true (case (a)),
- E is d-separated from D by A: false.

Exercise 2: Markov Chain

We have the following Markov Chain:



a) Write the joint probability distribution associated to this Markov Chain.

$$p(A, B, C, D) = \frac{1}{Z}\psi_{A,B}(A, B)\psi_{B,C}(B, C)\psi_{C,D}(C, D)$$

b) Each variable can take value 0 or 1, and we want to express that it is 9 times more probable that neighboring variables have equal values than they have different value. Give the potential functions of this Markov Chain.

All three potential functions are the same:

V_1 \ V_2	0	1
0	9	1
1	1	9

Notice that the values need not be normalized in any way.

c) Compute the probability distributions $p(A)$ and $p(C)$.

μ_α and μ_β can be calculated recursively:

$$\mu_\alpha(x_n) = \sum_{x_{n-1}} \psi_{n-1,n}(x_{n-1}, x_n)\mu_\alpha(x_{n-1})$$
$$\mu_\beta(x_n) = \sum_{x_{n+1}} \psi_{n,n+1}(x_n, x_{n+1})\mu_\beta(x_{n+1})$$

With our potentials this yields:

- $\mu_\alpha(A) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (initialization, optional),
- $\mu_\alpha(B) = \begin{pmatrix} \sum_A \mu_\alpha(A)\psi_{A,B}(A, 0) \\ \sum_A \mu_\alpha(A)\psi_{A,B}(A, 1) \end{pmatrix} = \begin{pmatrix} 1 \times 9 + 1 \times 1 \\ 1 \times 1 + 1 \times 9 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$
- $\mu_\alpha(C) = \begin{pmatrix} \sum_B \mu_\alpha(B)\psi_{B,C}(B, 0) \\ \sum_B \mu_\alpha(B)\psi_{B,C}(B, 1) \end{pmatrix} = \begin{pmatrix} 10 \times 9 + 10 \times 1 \\ 10 \times 1 + 10 \times 9 \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$
- $\mu_\alpha(D) = \begin{pmatrix} \sum_C \mu_\alpha(C)\psi_{C,D}(C, 0) \\ \sum_C \mu_\alpha(C)\psi_{C,D}(C, 1) \end{pmatrix} = \begin{pmatrix} 100 \times 9 + 100 \times 1 \\ 100 \times 1 + 100 \times 9 \end{pmatrix} = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}$
- $\mu_\beta(D) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (initialization, optional),

- $\mu_\beta(C) = \begin{pmatrix} \sum_D \mu_\beta(D)\psi_{C,D}(0,D) \\ \sum_D \mu_\beta(D)\psi_{C,D}(1,D) \end{pmatrix} = \begin{pmatrix} 1\times 9 + 1\times 1 \\ 1\times 1 + 1\times 9 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$
- $\mu_\beta(B) = \begin{pmatrix} \sum_C \mu_\beta(C)\psi_{B,C}(0,C) \\ \sum_C \mu_\beta(C)\psi_{B,C}(1,C) \end{pmatrix} = \begin{pmatrix} 10\times 9 + 10\times 1 \\ 10\times 1 + 10\times 9 \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$
- $\mu_\beta(A) = \begin{pmatrix} \sum_B \mu_\beta(B)\psi_{A,B}(0,B) \\ \sum_B \mu_\beta(B)\psi_{A,B}(1,B) \end{pmatrix} = \begin{pmatrix} 100\times 9 + 100\times 1 \\ 100\times 1 + 100\times 9 \end{pmatrix} = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}$

Then we compute the normalization factor Z at any point, for example B:

$$Z = \sum_B \mu_\alpha(B).\mu_\beta(B) = 2000$$

Finally we can compute the marginal distributions requested:

$$p(A) = \frac{1}{Z}.\mu_\alpha(A).\mu_\beta(A) = \frac{1}{2000}\begin{pmatrix} 1\times 1000 \\ 1\times 1000 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

$$p(C) = \frac{1}{Z}.\mu_\alpha(C).\mu_\beta(C) = \frac{1}{2000}\begin{pmatrix} 100\times 10 \\ 100\times 10 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

The assumptions were only that neighboring nodes should be equal. The marginal on A and C both say that we have no idea on their value. That was to be expected.

d) Now, we observe that D is 1, recompute the distributions over A and C: $p(A \mid [D=1])$ and $p(C \mid [D=1])$.

We've learned that we could compute marginal distributions by decomposing the inference into messages to be passed between nodes.

How can we adapt this mecanism to observations?

If the chain contained only C and D, we would have:

$$p(C \mid [D=1]) = \frac{1}{Z'}\begin{pmatrix} \psi_{C,D}(0,1) \\ \psi_{C,D}(1,1) \end{pmatrix}$$

This can be written in the same message passing form:

$$p(C \mid [D=1]) = \frac{1}{Z'}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.\begin{pmatrix} \sum_D \mu'_\beta(D)\psi_{C,D}(0,D) \\ \sum_D \mu'_\beta(D)\psi_{C,D}(1,D) \end{pmatrix}$$

with $\mu'_\beta(D) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Actually, you just have to replace the $\mu_*(X)$ with a Dirac in order to account for an observation of the value of X (and recompute the normalization factor):

- $\mu_\alpha(A) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

- $\mu_\alpha(B) = \begin{pmatrix} \sum_A \mu_\alpha(A)\psi_{A,B}(A,0) \\ \sum_A \mu_\alpha(A)\psi_{A,B}(A,1) \end{pmatrix} = \begin{pmatrix} 1\times 9 + 1\times 1 \\ 1\times 1 + 1\times 9 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$
- $\mu_\alpha(C) = \begin{pmatrix} \sum_B \mu_\alpha(B)\psi_{B,C}(B,0) \\ \sum_B \mu_\alpha(B)\psi_{B,C}(B,1) \end{pmatrix} = \begin{pmatrix} 10\times 9 + 10\times 1 \\ 10\times 1 + 10\times 9 \end{pmatrix} = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$
- $\mu_\alpha(D) = \begin{pmatrix} \sum_C \mu_\alpha(C)\psi_{C,D}(C,0) \\ \sum_C \mu_\alpha(C)\psi_{C,D}(C,1) \end{pmatrix} = \begin{pmatrix} 100\times 9 + 100\times 1 \\ 100\times 1 + 100\times 9 \end{pmatrix} = \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}$
- $\mu'_\beta(D) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (observation),
- $\mu'_\beta(C) = \begin{pmatrix} \sum_D \mu'_\beta(D)\psi_{C,D}(0,D) \\ \sum_D \mu'_\beta(D)\psi_{C,D}(1,D) \end{pmatrix} = \begin{pmatrix} 0\times 9 + 1\times 1 \\ 0\times 1 + 1\times 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$
- $\mu'_\beta(B) = \begin{pmatrix} \sum_C \mu'_\beta(C)\psi_{B,C}(0,C) \\ \sum_C \mu'_\beta(C)\psi_{B,C}(1,C) \end{pmatrix} = \begin{pmatrix} 1\times 9 + 9\times 1 \\ 1\times 1 + 9\times 9 \end{pmatrix} = \begin{pmatrix} 18 \\ 82 \end{pmatrix}$
- $\mu'_\beta(A) = \begin{pmatrix} \sum_B \mu'_\beta(B)\psi_{A,B}(0,B) \\ \sum_B \mu'_\beta(B)\psi_{A,B}(1,B) \end{pmatrix} = \begin{pmatrix} 18\times 9 + 82\times 1 \\ 18\times 1 + 82\times 9 \end{pmatrix} = \begin{pmatrix} 244 \\ 756 \end{pmatrix}$

As above, we can also compute $Z' = 1000$ and then:

$$p(A \mid [D=1]) = \frac{1}{Z'}.\mu_\alpha(A).\mu'_\beta(A) = \frac{1}{1000}\begin{pmatrix} 1\times 244 \\ 1\times 756 \end{pmatrix} = \begin{pmatrix} 0.244 \\ 0.756 \end{pmatrix}$$

$$p(C \mid [D=1]) = \frac{1}{Z'}.\mu_\alpha(C).\mu'_\beta(C) = \frac{1}{1000}\begin{pmatrix} 100\times 1 \\ 100\times 9 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix}$$

Now, we know that node D equals 1 and we see it has become more probable for A and C to be equal to 1 (the more for C which is nearer D than A). At least the result makes sense.

e) Compute $p(C \mid [A=0], [D=1])$.

With the same way, we can recompute μ'_α (which is symmetric to μ_β):

- $\mu'_\alpha(A) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,
- $\mu'_\alpha(B) = \begin{pmatrix} \sum_A \mu'_\alpha(A)\psi_{A,B}(A,0) \\ \sum_A \mu'_\alpha(A)\psi_{A,B}(A,1) \end{pmatrix} = \begin{pmatrix} 1\times 9 + 0\times 1 \\ 1\times 1 + 0\times 9 \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \end{pmatrix}$
- $\mu'_\alpha(C) = \begin{pmatrix} \sum_B \mu'_\alpha(B)\psi_{B,C}(B,0) \\ \sum_B \mu'_\alpha(B)\psi_{B,C}(B,1) \end{pmatrix} = \begin{pmatrix} 9\times 9 + 1\times 1 \\ 9\times 1 + 1\times 9 \end{pmatrix} = \begin{pmatrix} 82 \\ 18 \end{pmatrix}$
- $\mu'_\alpha(D) = \begin{pmatrix} \sum_C \mu'_\alpha(C)\psi_{C,D}(C,0) \\ \sum_C \mu'_\alpha(C)\psi_{C,D}(C,1) \end{pmatrix} = \begin{pmatrix} 82\times 9 + 18\times 1 \\ 82\times 1 + 18\times 9 \end{pmatrix} = \begin{pmatrix} 756 \\ 244 \end{pmatrix}$

Now $Z'' = 244$ and:

$$p(C \mid [D=1]) = \frac{1}{Z''}.\mu'_\alpha(C).\mu'_\beta(C) = \frac{1}{244}\begin{pmatrix} 82\times 1 \\ 18\times 9 \end{pmatrix} \approx \begin{pmatrix} 0.336 \\ 0.664 \end{pmatrix}$$

It would be the reverse for B: $\begin{pmatrix} 0.664 \\ 0.336 \end{pmatrix}$. It is not exactly $\frac{2}{3}$. Actually, with a longer chain, both μ'_α and μ'_β would (exponentially) converge to uniforms as we consider node further from their origin. Therefore for a long chain, the probability will come from $\begin{pmatrix} 100\% \\ 0\% \end{pmatrix}$ to rest at the uniform $\begin{pmatrix} 50\% \\ 50\% \end{pmatrix}$ before setting to $\begin{pmatrix} 0\% \\ 100\% \end{pmatrix}$. In order to "straighten" the values, we could lower the probability of being different from neighboring nodes.

Note that the known nodes are at the boundary of our chain. If it was not the case, the d-separation property would have allowed us to split the chain in two independent subchains having both a copy of the observed variable as the new boundary.