

Weekly Exercises 8

Room: 02.09.023

Monday, 03.07.2017, 12:15-14:00

Submission deadline: Wednesday, 28.06.2017, Room 02.09.023

Primal-Dual Methods

(6 Points)

Exercise 1 (2 Points). Consider the optimization problem

$$\min_{x \in \mathbb{R}^n} g(x) + \sum_{i=1}^k f_i(K_i x), \quad (1)$$

with $g : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$, $f_i : \mathbb{R}^{m_i} \rightarrow \bar{\mathbb{R}}$ closed, proper, convex and $K_i : \mathbb{R}^n \rightarrow \mathbb{R}^{m_i}$ linear. Assume that g and all f_i are *simple* in the sense that their proximal mapping

$$\text{prox}_{\tau f_i}(y) := \operatorname{argmin}_{x \in \mathbb{R}^{m_i}} f_i(x) + \frac{1}{2\tau} \|x - y\|^2,$$

can be efficiently computed. Explain how (1) can be solved with PDHG and write down the explicit update equations.

Hint: Stack the individual K_i into a single matrix K .

Solution. The optimization problem (1) can be rewritten in the standard form as

$$\min_{x \in \mathbb{R}^n} g(x) + f(Kx), \quad (2)$$

where $K = \begin{bmatrix} K_1 \\ \vdots \\ K_k \end{bmatrix}$, and $f(z_1, \dots, z_k) = \sum_{i=1}^k f_i(z_i)$. The PDHG updates are given

by:

$$\begin{aligned} x^{t+1} &= \text{prox}_{\tau g}(x^t - \tau \sum_{i=1}^k K_i^\top y_i^t), \\ y_i^{t+1} &= \text{prox}_{\sigma f_i^*}(y_i^t + \sigma K_i(2x^{t+1} - x^t)), \text{ for } 1 \leq i \leq k. \end{aligned} \quad (3)$$

Exercise 2 (4 Points). Recall the energy from the cartooning programming exercise on the second exercise sheet:

$$E(u) = \langle u, f \rangle + \sum_{j=1}^n \delta\{u(:, j) \in \Delta^k\} + \alpha \sum_{i=1}^k \|Du(i, :)\|_1. \quad (4)$$

Write down the explicit update equations for solving (4) with a) PDHG and b) ADMM. Make sure to apply the algorithms in such a way that each subproblem has a simple, closed form solution (such as orthogonal projection onto the simplex, solving a linear system or proximity operator of the ℓ_1 -norm).

Solution. Let $\text{proj}_C(x) = \text{argmin}_{y \in C} \|x - y\|$ denote the orthogonal projection of x onto the set C .

1. PDHG solution.

$$\begin{aligned}\tilde{u}(i, :)^{t+1} &= u(i, :)^t - \tau(D^\top q(i, :)^t + f(i, :)), \\ u(:, j)^{t+1} &= \text{proj}_{\Delta_k}(\tilde{u}(:, j)^t), \\ q^{t+1}(i, :) &= \text{proj}_{[-\alpha, \alpha]^n}(q(i, :)^t + \sigma D(2u(i, :)^{t+1} - u(i, :)^t)).\end{aligned}\tag{5}$$

2. ADMM solution. We consider the augmented Lagrangian to (4)

$$\begin{aligned}L_a(u, v_1, v_2, q_1, q_2) &= \langle u, f \rangle + \sum_{j=1}^n \delta\{v_1(:, j) \in \Delta^k\} + \alpha \sum_{i=1}^k \|v_2(i, :)\|_1 + \\ &\sum_{i=1}^k \langle q_2(i, :), Du(i, :) - v_2(i, :)\rangle + \frac{\rho}{2} \|Du(i, :) - v_2(i, :)\|^2 + \langle q_1, u - v_1 \rangle + \frac{\rho}{2} \|u - v_1\|^2.\end{aligned}\tag{6}$$

The ADMM updates are:

$$\begin{aligned}u^{t+1}(i, :) &= (I + D^\top D)^{-1}(v_1(i, :) + D^\top v_2(i, :) - \frac{1}{\rho}(f(i, :) + q_1(i, :) + D^\top q_2(i, :))), \\ v_1^{t+1}(:, j) &= \text{proj}_{\Delta_k}(u^{t+1}(:, j) + q_1^t(:, j)/\rho), \\ v_2^{t+1}(i, :) &= \text{prox}_{(\alpha/\rho)\|\cdot\|_1}(Du(i, :)^{t+1} + q_2(i, :)/\rho), \\ q_1^{t+1} &= q_1^t + \rho(u - v_1), \\ q_2^{t+1}(i, :) &= q_2^t + \rho(Du(i, :) - v_2(i, :)).\end{aligned}\tag{7}$$

Programming

(5 Bonus)

Exercise 3. Implement one of the methods you derived in exercise 2 using MATLAB. For PDHG, choose the step sizes s and t such that they satisfy $st > \|K\|_{\text{spec}}^2$.

The deadline for handing in the programming exercise is **July 5th, 2017, 23:59pm**.