

## Weekly Exercises 7

Room 01.09.014

Friday, 10.6.2016, 09:00-11:00

Submission deadline: Wednesday, 8.6.2016, 14:15, Room: 02.09.023

### Theory: (12+2 Points)

**Exercise 1** (6 Points). Let  $F$  and  $G$  be proper, closed, convex functions, let  $E(u) := G(u) + F(Ku)$  have a minimizer, and let there be a  $u \in \text{ri}(\text{dom}(G))$  such that  $Ku \in \text{ri}(\text{dom}(F))$ . We define

$$PD(u, p) = G(u) + \langle Ku, p \rangle - F^*(p).$$

Show that there exists a point  $(\tilde{u}, \tilde{p})$  such that the (unrestricted) primal-dual gap

$$\mathcal{G}(u, p) = \sup_{p'} PD(u, p') - \inf_{u'} PD(u', p)$$

is zero, i.e.  $\mathcal{G}(\tilde{u}, \tilde{p}) = 0$ .

**Solution.** We find that

$$\begin{aligned} \inf_{u'} PD(u', p) &= \inf_{u'} G(u') + \langle Ku', p \rangle - F^*(p) \\ &= - \sup_{u'} \langle u', -K^*p \rangle - G(u') + F^*(p) \\ &= -G^*(-K^*p) - F^*(p) \end{aligned}$$

and similarly

$$\begin{aligned} \inf_{p'} PD(u, p') &= \sup_{p'} G(u) + \langle Ku, p' \rangle - F^*(p') \\ &= G(u) + F^{**}(Ku) \\ &= G(u) + F(Ku). \end{aligned}$$

Therefore

$$\mathcal{G}(u, p) = G(u) + F(Ku) - (-G^*(-K^*p) - F^*(p)).$$

The conditions on the relative interior imply that Fechel's Duality Theorem can be applied such that  $\inf_u \sup_p \mathcal{G}(u, p) = 0$ . Left to show is that  $\inf_u \sup_p$  actually is a  $\min_u \max_p$ . By the assumptions of the exercise, a minimizer  $\tilde{u}$  exists. Since the sum rule is applicable, we know that

$$0 \in \partial G(\tilde{u}) + K^* \partial F(K\tilde{u}),$$

i.e. there exist a  $\tilde{p} \in \partial F(K\tilde{u})$  such that  $-K^*\tilde{p} \in \partial G(\tilde{u})$ . By the identity relating the subdifferential of a function to the subdifferential of its dual it holds that  $K\tilde{u} \in \partial F^*(\tilde{p})$  and  $\tilde{u} \in \partial G^*(-K^*\tilde{p})$ . This means

$$0 = K\tilde{u} + (-K\tilde{u}) \in \partial F^*(\tilde{p}) + (-K\partial G^*(-K^*\tilde{p})),$$

and implies that  $\tilde{p}$  minimizes  $G^*(-K^*p) + F^*(p)$  or, equivalently, maximizes  $-G^*(-K^*p) - F^*(p)$ .

**Exercise 2** (2 Points). Prove that the algorithm

$$\begin{aligned} u^{k+1} &= \text{prox}_{\tau G}(u^k - \tau K^* \bar{p}^k), \\ p^{k+1} &= \text{prox}_{\sigma F^*}(p^k + \sigma K u^{k+1}), \\ \bar{p}^{k+1} &= 2p^{k+1} - p^k. \end{aligned} \tag{PDHG*}$$

converges, and the limit of the  $u^k$  is a minimizer of  $G(u) + F(Ku)$  (with the same assumptions on  $F$ ,  $G$ , and  $K$  as in the previous exercise).

*Hint: Show that (PDHG\*) is equivalent to an algorithm we discussed in the lecture applied to a reformulated problem!*

**Solution.** By Fenchel's Duality Theorem, computing  $\min_u G(u) + F(Ku)$  is the same as computing  $-\min_p G^*(-K^*p) + F^*(p)$ , where the minimizers of the first and the second problem are related via  $p \in \partial F(Ku)$ . By replacing  $G^*(-K^*p) = \sup_u \langle -K^*p, u \rangle - G(u)$ , we find

$$\begin{aligned} \min_u G(u) + F(Ku) &= -\min_p G^*(-K^*p) + F^*(p) \\ &= -\min_p \max_u F^*(p) + \langle -Ku, p \rangle - G(u) \end{aligned}$$

Applying the usual (PDHG) algorithm to the  $\min_p \max_u$  in the second line yields (PDHG\*) for which we established the convergence in the lecture.

**Exercise 3** (6 Points). Derive explicit formulas for the update equations for solving the following problems with the (PDHG) algorithm:

- TV- $\ell^1$  denoising, i.e.

$$\min_u \|u - f\|_1 + \alpha \|Ku\|_{2,1}$$

for a linear operator  $K : \mathbb{R}^{n \times m \times 3} \rightarrow \mathbb{R}^{nm \times 2c}$  (e.g. being the finite difference color gradient operator).

- Image segmentation, i.e.

$$\min_u \iota_{\Delta}(u) + \iota_{\geq 0}(u) + \langle u, f \rangle + \alpha \|Ku\|_{2,1}$$

for a linear operator  $K : \mathbb{R}^{n \times m \times 3} \rightarrow \mathbb{R}^{nm \times 2}$  (e.g. being the finite difference gradient operator).

- Sparse recovery, i.e.

$$\min_u \frac{1}{2} \|Au - f\|^2 + \alpha \|u\|_1$$

for  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  being a linear operator.

**Solution.** • Dual: Project each column onto the  $\ell^2$  ball. Primal: Shrinkage+substitution - compare sheet 3 exercise 3.

- Dual 1: Project each column onto the  $\ell^2$  ball, dual 2: prox for  $F^*(q) = q$ , primal: nonnegative + linear
- Dual  $\ell^2$ -squared prox, primal: shrinkage.

## Programming: TV- $\ell^1$ -denoising revisited (12 Points)

**Exercise 4** (12 Points). Consider the salt-and-pepper removal algorithm from Exercise 4 on Sheet 3 again, i.e.

$$u^* = \arg \min_u \|u - f\|_1 + \alpha \|Du\|_{2,1}, \quad (1)$$

where  $K$  is a finite difference color gradient operator.

On Sheet 3 you implemented subgradient descent. Now implement the (PDHG) algorithm to compute the argument that minimizes (1). Plot the decay of energy of the subgradient descent and the (PDHG) method for at least two different choices of stepsizes  $\tau$  and  $\sigma$  on the `fish_saltpepper.png` image.

*Hint: To make sure the step size restriction of (PDHG) is met, the MATLAB function "normest" can be useful.*