Suggested Homework

Nonlinear Multiscale Methods for Image and Signal Analysis

Exercise 1. Let $q \in \mathbb{R}^{n\times m}$. Convince yourself that the projection $\hat{p}$ of q onto the $\ell^{2,\infty}$ ball of radius one,

$$B_{\ell^{2,\infty}} = \left\{ p \in \mathbb{R}^{n\times m} \mid \sqrt{\sum_j (p_{ij})^2} \leq 1, \ \forall i \right\},$$

i.e.

$$\hat{p} = \arg \min_{p \in B_{\ell^{2,\infty}}} \|p - q\|_F^2,$$

is given by

$$\hat{p}_{ij} = \frac{q_{ij}}{\max\left(\sqrt{\sum_j (q_{ij})^2}, 1\right)}.$$

Proof. Let us use the notation $q_{i,:}$ for the i^{th} row of the matrix q. The projection problem can also be written as

$$\begin{aligned}\hat{p} &= \arg \min_{p \in B_{\ell^{2,\infty}}} \|p - q\|_F^2 \\ &= \arg \min_{p \in \mathbb{R}^{n\times m},\ \|p_{i,:}\|_2 \leq 1} \sum_i \|p_{i,:} - q_{i,:}\|_2^2.\end{aligned}$$

As we can see, the minimization problem decouples in the rows of q, such that $p_{i,:}$ is given by

$$\begin{aligned}\hat{p}_{i,:} &= \arg \min_{\|p_{i,:}\|_2 \leq 1} \|p_{i,:} - q_{i,:}\|_2^2 \\ &= \arg \min_{p_{i,:}} \|p_{i,:} - q_{i,:}\|_2^2 + \mathfrak{i}_{\|\cdot\|_2 \leq 1}(p_{i,:}),\end{aligned}$$

where $\mathfrak{i}_{\|\cdot\|_2 \leq 1}$ denotes the indicator function of the ℓ^2 unit ball. Although one might be able to "see" the solution right away, let us use subdifferential calculus. The optimality condition yields

$$2(\hat{p}_{i,:} - q_{i,:}) + z = 0$$

for some $z \in \partial \mathfrak{i}_{\|\cdot\|_2 \leq 1}(\hat{p}_{i,:})$. Such a z must meet

$$\mathfrak{i}_{\|\cdot\|_2 \leq 1}(\rho) - \mathfrak{i}_{\|\cdot\|_2 \leq 1}(\hat{p}_{i,:}) - \langle z, \rho - \hat{p}_{i,:} \rangle \geq 0, \qquad \forall \rho,$$

or in other words

$$\begin{aligned} \|\hat{p}_{i,:}\|_2 &\leq 1, \qquad && \text{and} \\ \langle z, \hat{p}_{i,:} - \rho \rangle &\geq 0, \qquad && \forall \|\rho\|_2 \leq 1. \end{aligned}$$

It is easy to see that this implies $z = a\hat{p}_{i,:}$ for $a \geq 0$ as well as $a = 0$ if $\|\hat{q}_{i,:}\|_2 \neq 1$. Reinserting this finding into our optimality condition yields

$$(2 + a)\hat{p}_{i,:} + 2q_{i,:} = 0.$$

If we try to find a solution $\hat{p}_{i,:}$ with $\|\hat{p}_{i,:}\|_2 < 1$, we obtain $a = 0$ and finally a contradiction to $\|q_{i,:}\|_2 > 1$. Thus , $\|\hat{p}_{i,:}\|_2 = 1$ from which we obtain

$$\hat{p}_{i,:} = \frac{2}{2 + a} q_{i,:},$$

with a chose such that $\|\hat{p}_{i,:}\|_2 = 1$, i.e.

$$\hat{p}_{i,:} = \frac{q_{i,:}}{\|q_{i,:}\|_2},$$

which yields the assertion. $\square$