

Analysis of Three-Dimensional Shapes

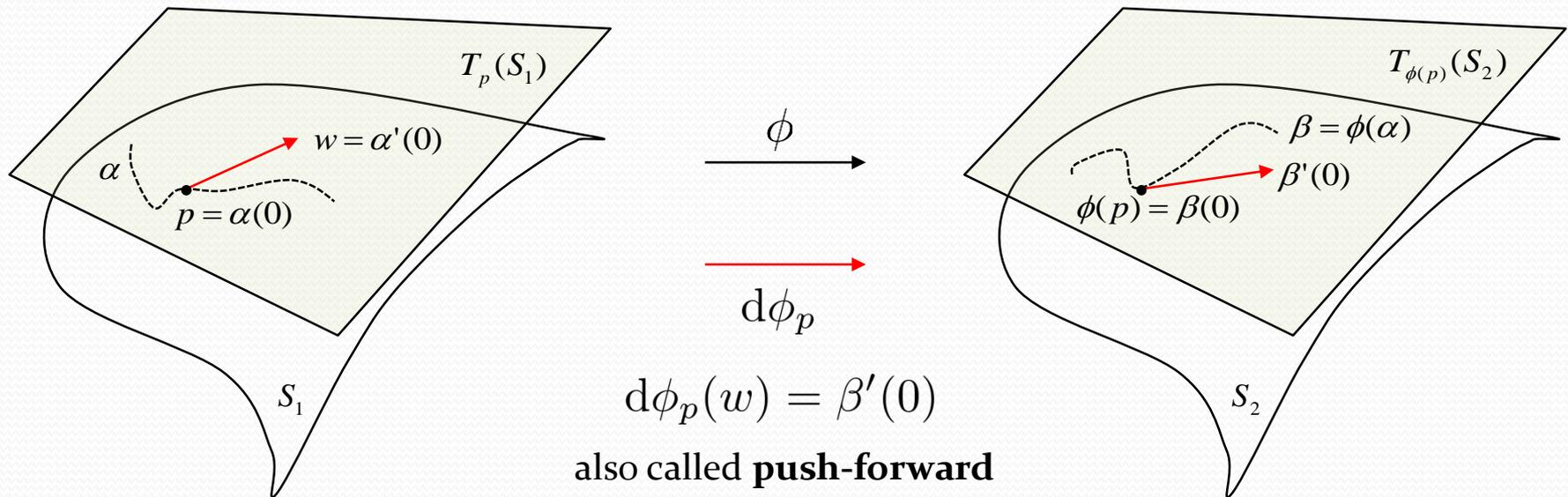
(IN2238, TU München, Summer 2015)

The Gradient
(11.05.2015)

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Wrap-up

After studying the fundamental properties of regular surfaces, we considered the case in which we are given **two** of them, and a differentiable **map** between them $\phi : V_1 \subset S_1 \rightarrow S_2$. We further defined the **differential** of this map.

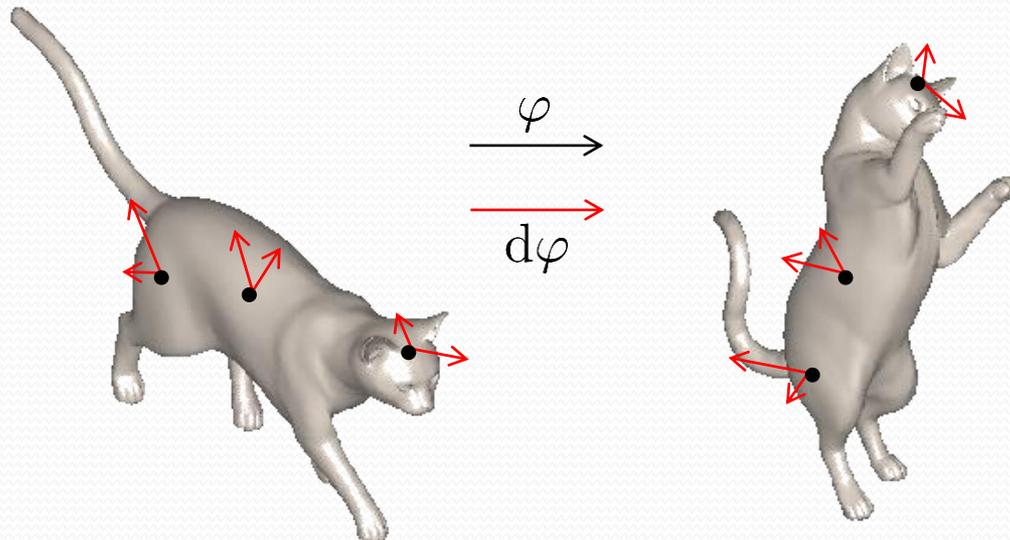


Wrap-up

We defined the key notion of **isometry**, that is a diffeomorphism $\varphi : S \rightarrow \bar{S}$ whose associated differential $d\varphi$ **preserves the inner product**:

$$\langle w_1, w_2 \rangle = \langle d\varphi_p(w_1), d\varphi_p(w_2) \rangle$$

for all $p \in S$ and all pairs of tangent vectors $w_1, w_2 \in T_p S$.



Wrap-up

We have also shown that **isometries preserve the first fundamental form**, and viceversa.

$$I_p(w) = \langle w, w \rangle = \langle d\varphi_p(w), d\varphi_p(w) \rangle = I_{\varphi(p)}(d\varphi_p(w))$$

As a key result, we have the **equivalence** of the two notions of isometry: distance-preserving and inner product-preserving.

$$I_p(w) = I_{\varphi(p)}(d\varphi_p(w)) \quad \Leftrightarrow \quad d(p, q) = d(\varphi(p), \varphi(q))$$

where we use our “natural” notion of distance, defined as:

$$d(p, q) = \inf_{\alpha: [0,1] \rightarrow S} \int_0^1 \sqrt{I_p(w(t))} dt$$

First fundamental form

We introduced the notion of first fundamental form on a regular surface as the quadratic function $I_p : T_p(S) \rightarrow \mathbf{R}$ given by

$$I_p((\alpha, \beta)) = \langle w, w \rangle = \|w\|^2$$

which, given a vector w in the tangent plane at p , simply computes its length.

In fact, we can generalize this function to take *two* arguments as follows:

$$I_p : T_p(S) \times T_p(S) \rightarrow \mathbf{R}$$

$$I_p((\alpha, \beta), (\gamma, \delta)) = \langle (\alpha, \beta), (\gamma, \delta) \rangle$$

The first fundamental form is a tool we use to compute **angles**, **lengths**, and **areas** on the surface.

First fundamental form

The first fundamental form can be conveniently rewritten as:

$$I_p((\alpha, \beta)) = \langle \alpha \mathbf{x}_u + \beta \mathbf{x}_v, \alpha \mathbf{x}_u + \beta \mathbf{x}_v \rangle = (\alpha \quad \beta) \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_g \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$
$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$$

Or, in case we regard it as the more general bilinear form, as:

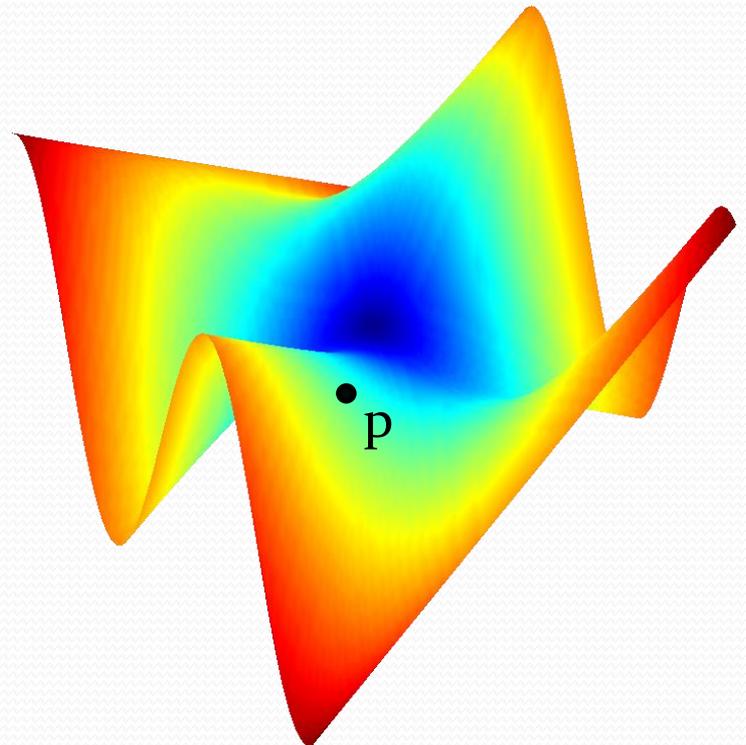
$$I_p((\alpha, \beta), (\gamma, \delta)) = \langle \alpha \mathbf{x}_u + \beta \mathbf{x}_v, \gamma \mathbf{x}_u + \delta \mathbf{x}_v \rangle = (\alpha \quad \beta) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

Gradient of a function

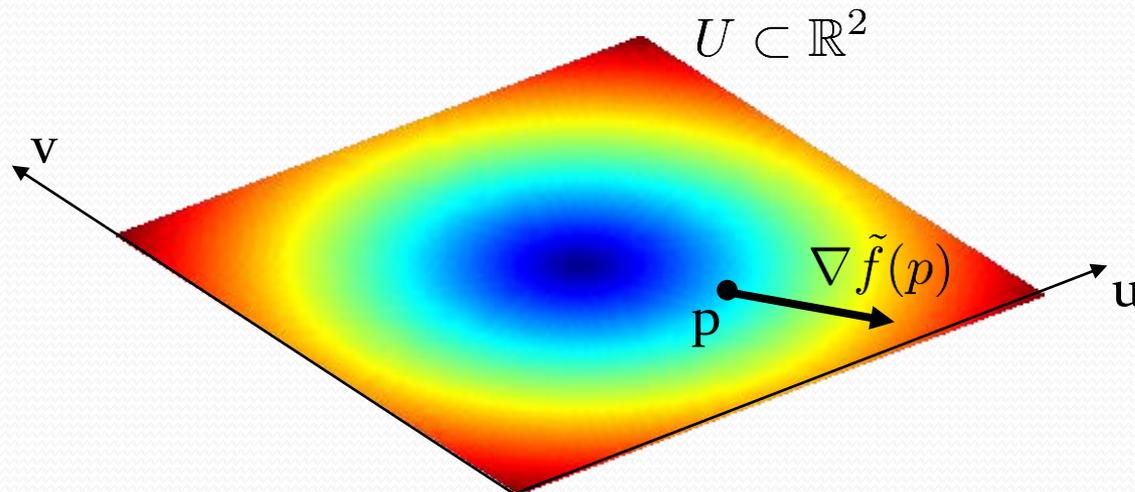
Consider a surface S with parametrization $\mathbf{x} : U \rightarrow S$
and a differentiable function $f : S \rightarrow \mathbb{R}$

We want to **define** the gradient $\nabla f(p)$
at a point $p \in S$.

As usual, we expect our definition
of gradient to be independent of
the parametrization function \mathbf{x} .



The gradient in \mathbb{R}^2



The gradient of a differentiable function $\tilde{f} : U \rightarrow \mathbb{R}$ is the vector field

$$\nabla \tilde{f}(p) = \begin{pmatrix} \frac{\partial \tilde{f}}{\partial u}(p) \\ \frac{\partial \tilde{f}}{\partial v}(p) \end{pmatrix}$$

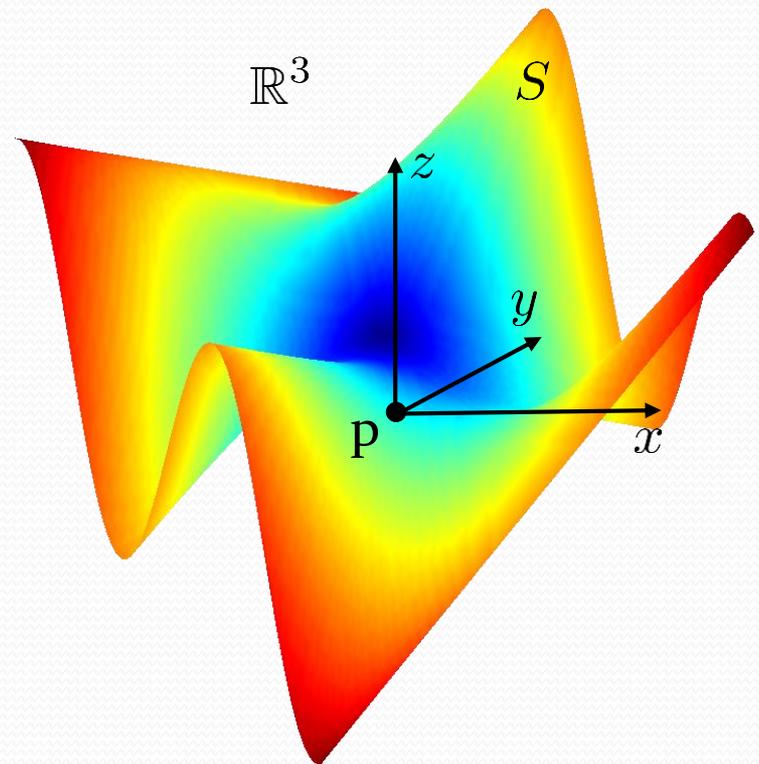
The gradient on a surface

Let now $f : S \rightarrow \mathbb{R}$ be a differentiable function on a regular surface $S \subset \mathbb{R}^3$.

Ideas how to define $\nabla f(p)$:

- Use the same formula as before, but in terms of x, y, z :

$$\nabla f(p) = \begin{pmatrix} \frac{\partial f}{\partial x}(p) \\ \frac{\partial f}{\partial y}(p) \\ \frac{\partial f}{\partial z}(p) \end{pmatrix}$$



Not a good choice, because we have **no information about f outside of S !**

The gradient on a surface

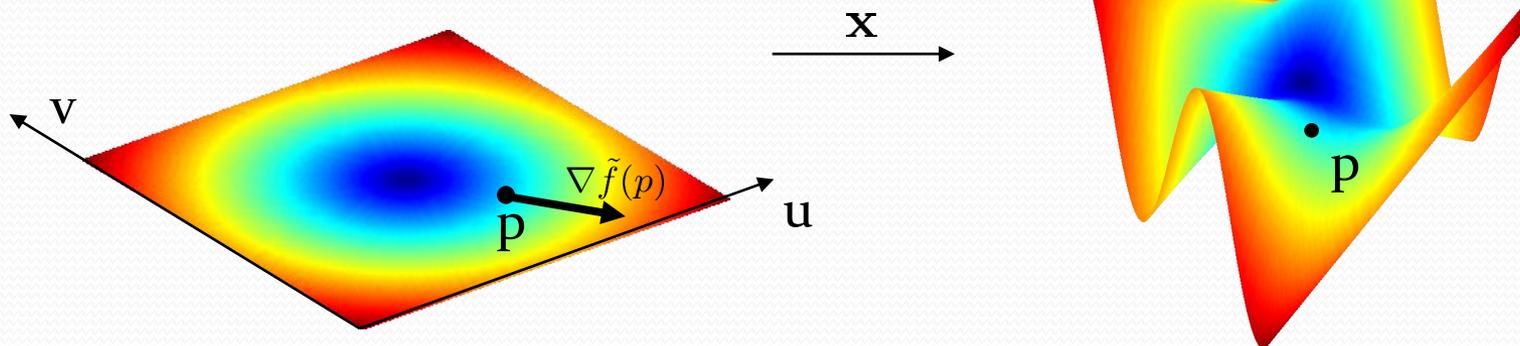
- Another possibility is to express our differentiable function f in terms of a parametrization \mathbf{x} :

$$\tilde{f}(u, v) = f(\mathbf{x}(u, v))$$

and then set:

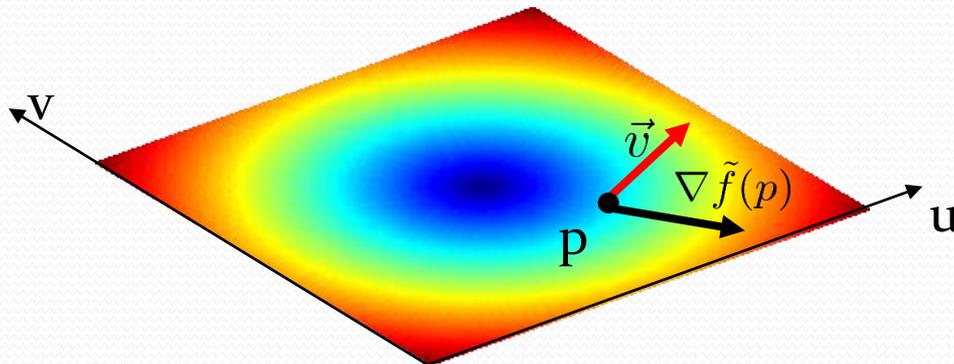
$$\nabla f(p) = \begin{pmatrix} \frac{\partial \tilde{f}}{\partial u}(p) \\ \frac{\partial \tilde{f}}{\partial v}(p) \end{pmatrix}$$

Depends on the choice of the parametrization!



The gradient on a surface

- Instead, let us try to interpret the **geometric meaning** of the gradient:
 - The vector that points in the **direction of steepest increase** of f
 - Its length measures the strength of increase
 - We have a relationship with the **directional derivative**:



$$\begin{aligned}df_p(\vec{v}) &= \lim_{h \rightarrow 0} \frac{\tilde{f}(p + h\vec{v}) - \tilde{f}(p)}{h} \\&= \frac{d}{dh} \tilde{f}(p + h\vec{v})|_{h=0} \\&= \langle \nabla \tilde{f}, \vec{v} \rangle\end{aligned}$$

directional derivative of f at p , along direction v

Representation theorem

In fact, the gradient of any differentiable function f can be **defined** as the **unique vector field** ∇f such that the relationship holds:

$$\langle \nabla f, \vec{v} \rangle = df_p(\vec{v}) \quad \text{where } df_p(\vec{v}) : T_p S \rightarrow \mathbb{R}$$

This is an application of the **Riesz representation theorem**:

H
Hilbert space
(vector space
with an inner
product)

$H^* = \{ \phi : H \rightarrow \mathbb{R} \mid \phi \text{ continuous, linear} \} = \{ df_p(\vec{v}) : T_p S \rightarrow \mathbb{R} \}$
dual space of H

Then, every $\phi \in H^*$ can be written *uniquely* as an inner product:

$$\phi(y) = \langle y, x \rangle \quad \forall y \in H \quad \Rightarrow \quad df_p(\vec{v}) = \langle \vec{v}, x \rangle \quad \forall \vec{v} \in H$$

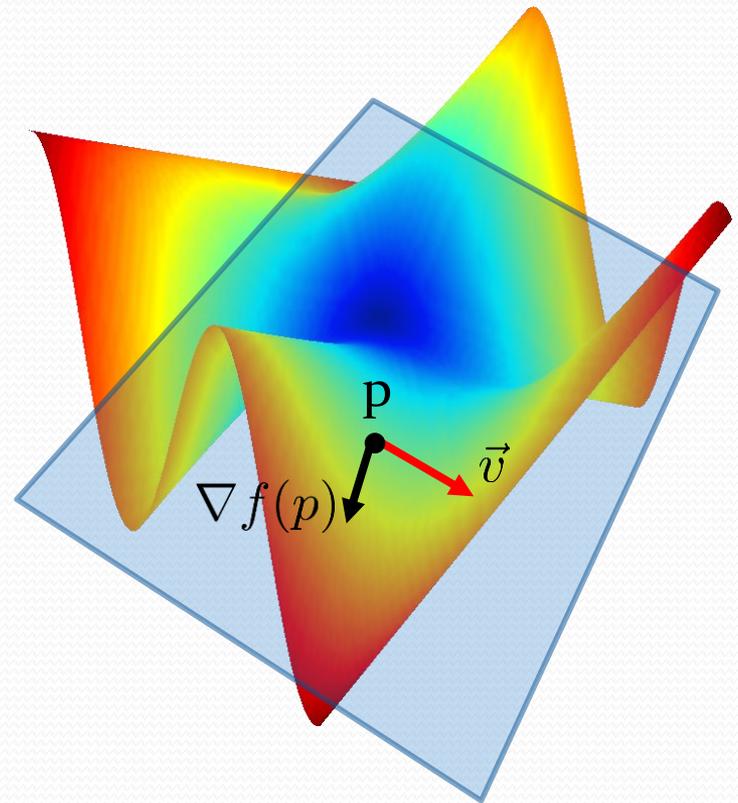
And one can then define the gradient as the *unique* $x \equiv \nabla f : S \rightarrow T_p S$

The gradient on a surface

We define the gradient $\nabla f(p) \in T_p S$ by means of the inner product:

$$I_p(\nabla f, \vec{v}) = df_p(\vec{v}) \quad \forall \vec{v} \in T_p S$$

Thus, if we are able to write down $df_p(\vec{v})$, we can “solve for” ∇f .

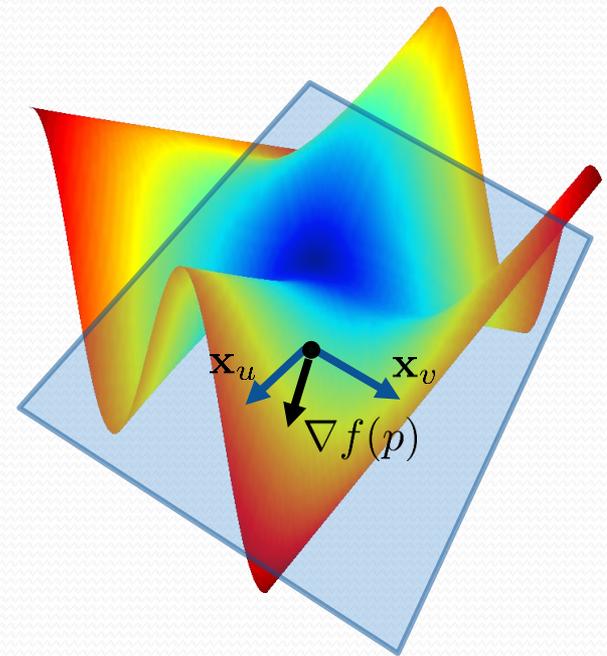


The gradient in local coordinates

Also note that, since the gradient is a member of $T_p S$, we can express it in the local basis:

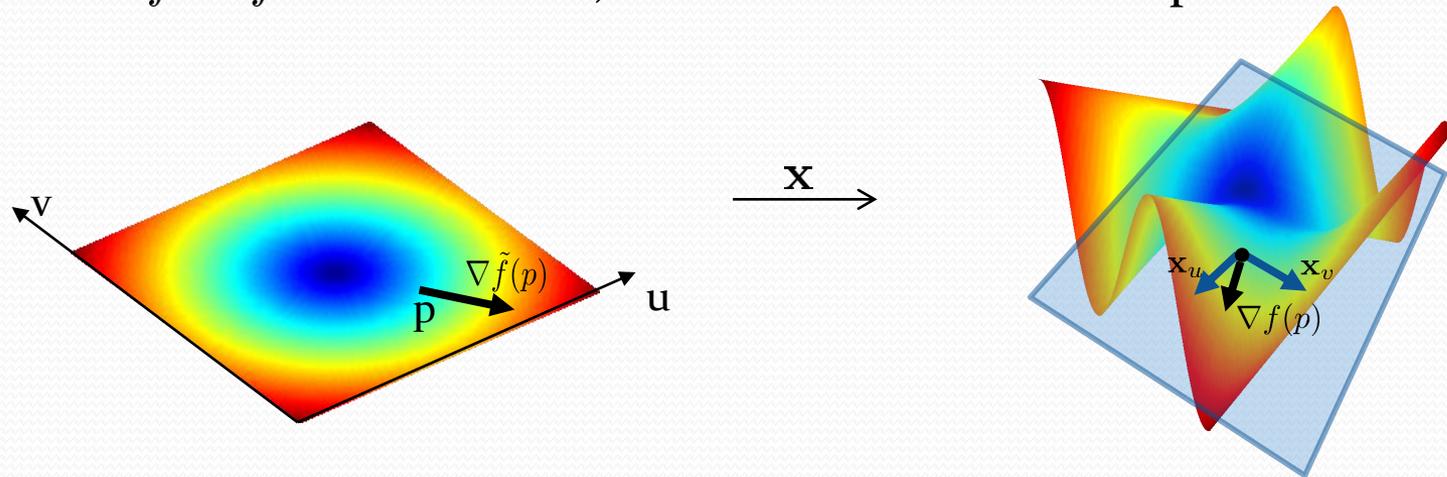
$$\nabla f(p) = f_1 \mathbf{x}_u + f_2 \mathbf{x}_v = D\mathbf{x} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Hence, knowledge of the coefficients f_1, f_2 corresponds to knowledge of $\nabla f(p)$.



The gradient in local coordinates

It turns out that the coefficients f_1, f_2 can be obtained by considering the gradient of $\tilde{f} = f \circ \mathbf{x} : U \rightarrow \mathbb{R}$, which is defined on the parameter domain.



Since we have $f(\mathbf{x}(u, v)) = \tilde{f}(u, v)$, we also have that the change of f in the direction $\mathbf{x}_u/\mathbf{x}_v$ corresponds to the change of \tilde{f} in the direction u/v :

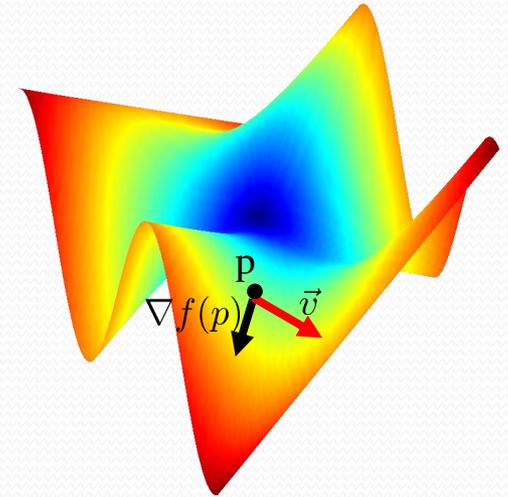
$$\frac{\partial \tilde{f}}{\partial u} = \frac{\partial (f \circ \mathbf{x})}{\partial u} \underset{\substack{\uparrow \\ \text{chain} \\ \text{rule}}}{=} \frac{\partial f}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial u} \underset{\substack{\uparrow \\ \text{Riesz}}}{=} df_p(\mathbf{x}_u)$$

Hence, we can compute the directional derivative df_p directly in parameter space!

The gradient in local coordinates

Now let $\vec{v} = v_1 \mathbf{x}_u + v_2 \mathbf{x}_v$, and apply df_p on both sides (note that df_p is linear):

$$\begin{aligned} df_p(\vec{v}) &= v_1 df_p(\mathbf{x}_u) + v_2 df_p(\mathbf{x}_v) \\ &= v_1 \frac{\partial \tilde{f}}{\partial u} + v_2 \frac{\partial \tilde{f}}{\partial v} \\ &= (\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{aligned}$$



On the other hand, we can also write:

$$df_p(\vec{v}) = I_p(\nabla f, \vec{v}) = (f_1 \quad f_2) g \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

This means $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = g^{-1} \nabla \tilde{f}$

Wrap-up

Thus, according to our definition of the gradient, we have to find the unique ∇f such that:

$$\langle \nabla f, \vec{v} \rangle = df_p(\vec{v})$$

Interestingly, by passing to local coordinates, we noticed that we can compute the directional derivative directly in U , as:

$$df_p(\vec{v}) = (\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

where $\tilde{f} = f \circ \mathbf{x}$

defined on
the surface S

defined on
the parameter
domain U

We can thus write $\langle \nabla f, \vec{v} \rangle = (\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

Wrap-up

Using the bilinear definition of first fundamental form, we can also write

$$\langle \nabla f, \vec{v} \rangle = I_p(\nabla f, \vec{v}) = (f_1 \quad f_2) g \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Together with the last equation from the previous slide, we have

$$(\nabla \tilde{f})^T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (f_1 \quad f_2) g \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

And thus we finally obtain:

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = g^{-1} \nabla \tilde{f}$$

Expression in local coordinates

The expression

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = g^{-1} \nabla \tilde{f}$$

is giving the gradient coefficients w.r.t. a basis in $T_p S$, hence it is said to be given **in local coordinates**.

It is quite common to see the alternative notation:

$$(\nabla f)^i = \sum_{j=1}^n g^{ij} \frac{\partial f}{\partial x^j} = \boxed{g^{ij} \frac{\partial f}{\partial x^j}}$$

- Where we write $g^{ij} \equiv (g^{-1})_{ij}$, and in our case $n = 2$
- The last expression follows Einstein summation convention, which sums over repeated indices

In order to obtain a vector $\nabla f \in \mathbb{R}^3$, we simply have to write:

$$\nabla f = f_1 \mathbf{x}_u + f_2 \mathbf{x}_v = D\mathbf{x} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = D\mathbf{x} g^{-1} \nabla \tilde{f}$$

Gradient norm

In some applications we may be interested in computing the **norm** of the gradient. Our final expression for the gradient is given by:

$$\nabla f = \mathbf{D}\mathbf{x} g^{-1} \nabla \tilde{f}$$

Computing its (squared) norm is straightforward:

$$\begin{aligned} \|\nabla f\|^2 &= \nabla f^\top \nabla f = \left(\mathbf{D}\mathbf{x} g^{-1} \nabla \tilde{f} \right)^\top \left(\mathbf{D}\mathbf{x} g^{-1} \nabla \tilde{f} \right) \\ &= \nabla \tilde{f}^\top g^{-1} \underbrace{\mathbf{D}\mathbf{x}^\top \mathbf{D}\mathbf{x}}_g g^{-1} \nabla \tilde{f} \\ &= \nabla \tilde{f}^\top g^{-1} \nabla \tilde{f} \end{aligned}$$

Example

Consider the function $f : \mathbb{S}^2 \setminus \{n\} \rightarrow \mathbb{R}$ that assigns, to each point on the unit sphere, its distance to the north pole n :

$$f(p) = d_{\mathbb{S}^2}(n, p)$$

We want to compute its gradient:

$$\nabla f = D_{\mathbf{x}} g^{-1} \nabla \tilde{f}$$

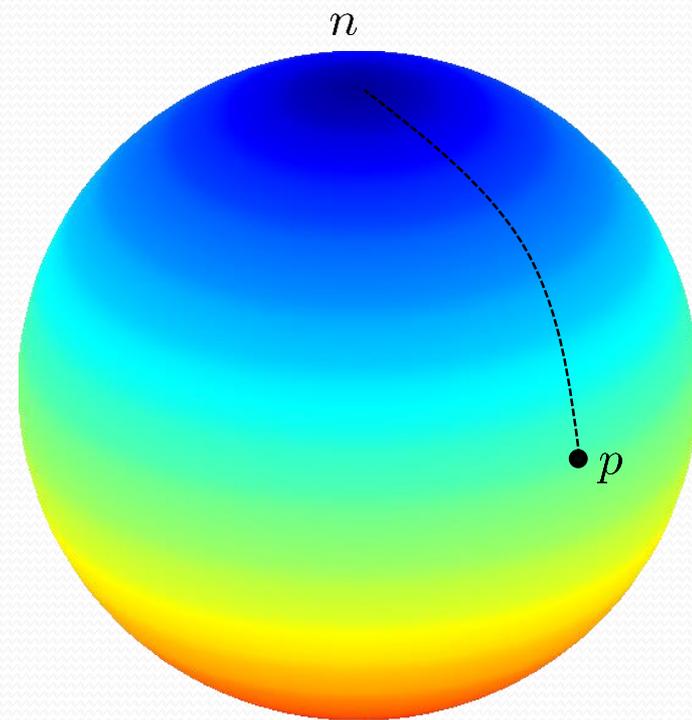
We will consider two different parametrizations:

$$\mathbf{x} : (0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}^3$$

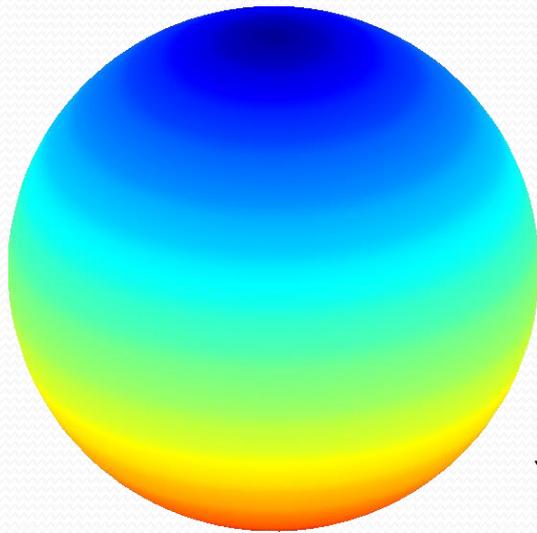
$$\mathbf{x}(u, v) = \begin{pmatrix} \cos(u) \cos(v) \\ \sin(u) \cos(v) \\ \sin(v) \end{pmatrix}$$

$$\mathbf{y} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$\mathbf{y}(\tilde{u}, \tilde{v}) = \frac{1}{\tilde{u}^2 + \tilde{v}^2 + 1} \begin{pmatrix} 2\tilde{u} \\ 2\tilde{v} \\ \tilde{u}^2 + \tilde{v}^2 - 1 \end{pmatrix}$$

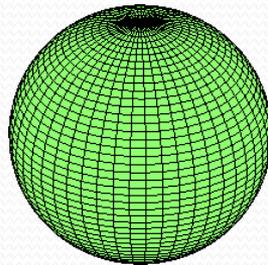


Example: first parametrization

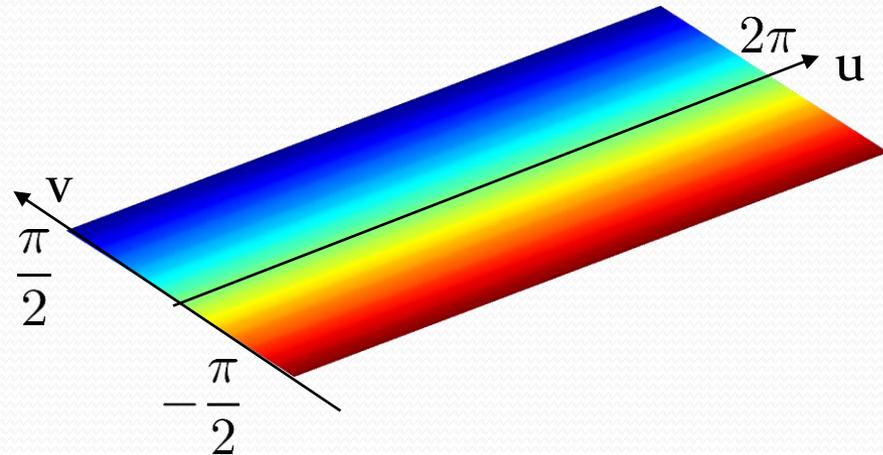
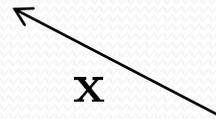


$$f(p) = d_{\mathbb{S}^2}(n, p)$$

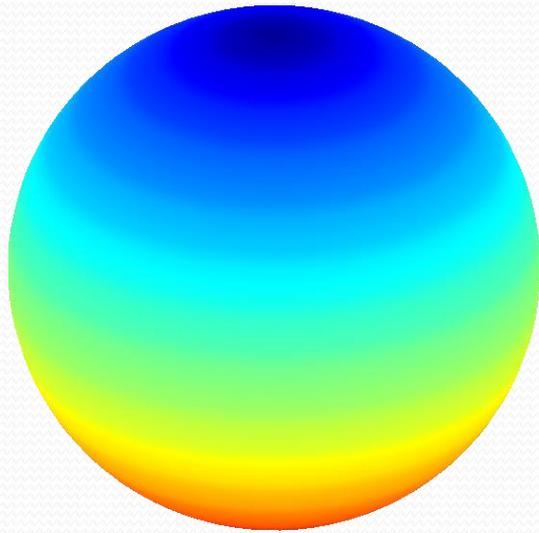
$$\tilde{f}(u, v) = \frac{\pi}{2} - v$$



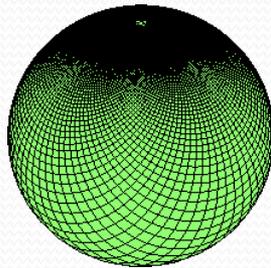
$$\mathbf{x} : (0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}^3$$
$$\mathbf{x}(u, v) = \begin{pmatrix} \cos(u) \cos(v) \\ \sin(u) \cos(v) \\ \sin(v) \end{pmatrix}$$



Example: second parametrization

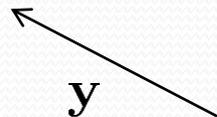


$$f(p) = d_{\mathbb{S}^2}(n, p)$$

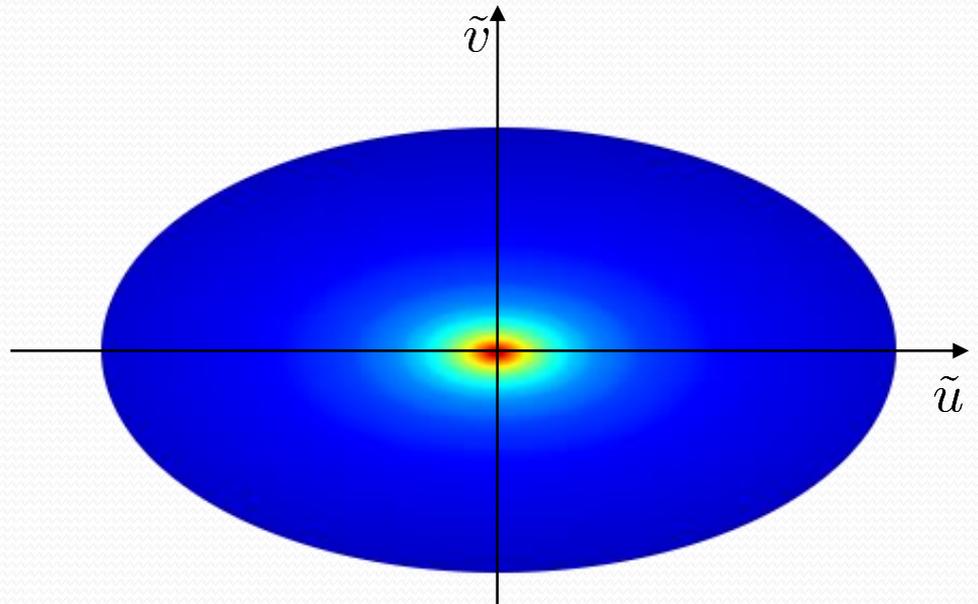


$$\mathbf{y} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

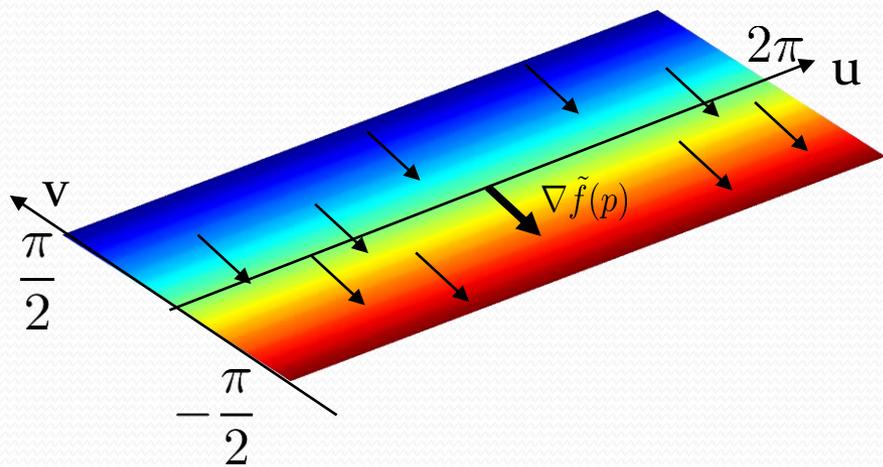
$$\mathbf{y}(\tilde{u}, \tilde{v}) = \frac{1}{\tilde{u}^2 + \tilde{v}^2 + 1} \begin{pmatrix} 2\tilde{u} \\ 2\tilde{v} \\ \tilde{u}^2 + \tilde{v}^2 - 1 \end{pmatrix}$$



$$\hat{f}(\tilde{u}, \tilde{v}) = -\arcsin\left(\frac{\tilde{u}^2 + \tilde{v}^2 - 1}{\tilde{u}^2 + \tilde{v}^2 + 1}\right)$$

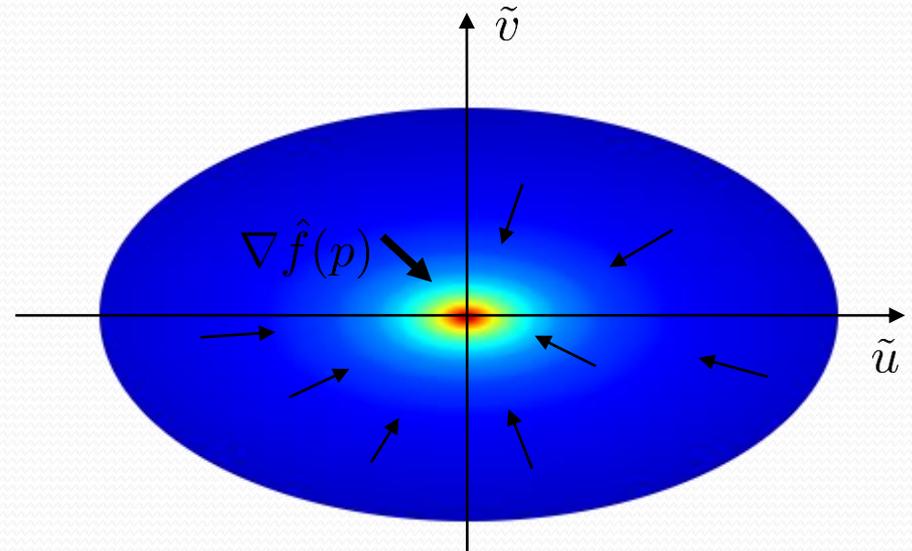


Example: gradients in \mathbb{R}^2



$$\tilde{f}(u, v) = \frac{\pi}{2} - v$$

$$\nabla \tilde{f}(u, v) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

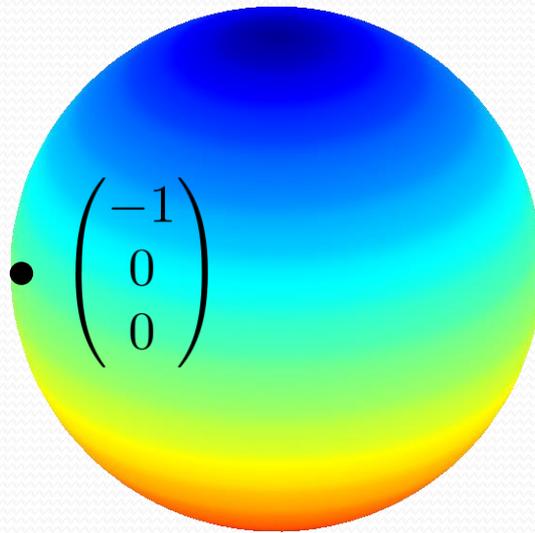


$$\hat{f}(\tilde{u}, \tilde{v}) = -\arcsin\left(\frac{\tilde{u}^2 + \tilde{v}^2 - 1}{\tilde{u}^2 + \tilde{v}^2 + 1}\right)$$

$$\nabla \hat{f}(\tilde{u}, \tilde{v}) = \frac{-2}{r(r^2 + 1)} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

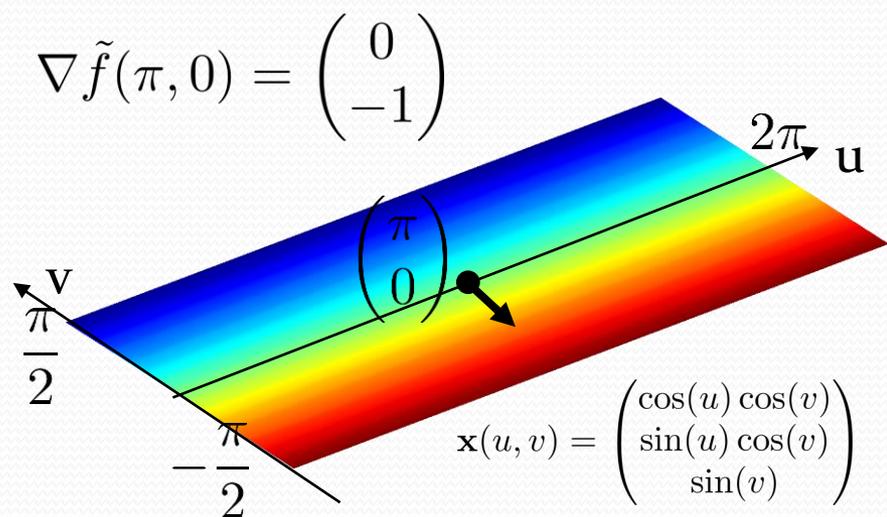
Clearly, the two parametrizations give rise to different gradients on each domain. For this reason, we said $\nabla f(p) = \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)$ is not a good definition for the gradient.

Example: gradient on the surface

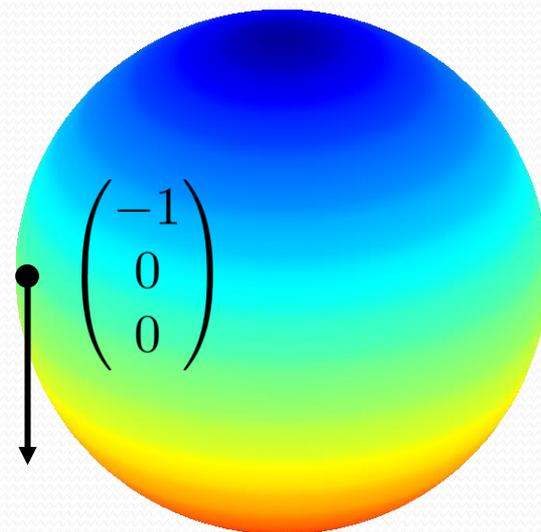


$$\nabla f(-1, 0, 0) = ?$$

First parametrization



\mathbf{x}



$$D\mathbf{x}(\pi, 0) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$$

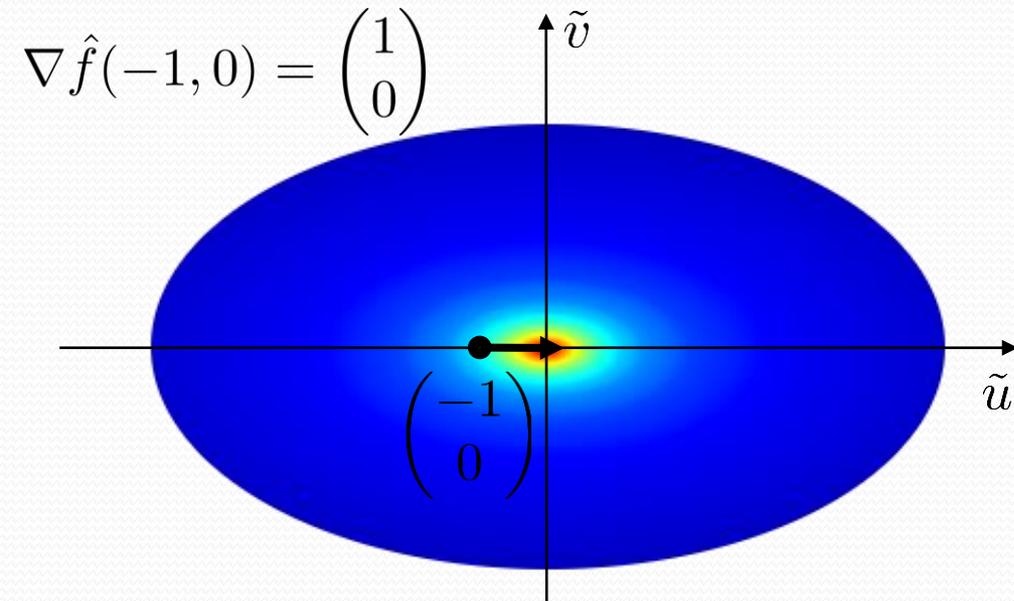
$$g_{\mathbf{x}}(\pi, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (g_{\mathbf{x}}(\pi, 0))^{-1}$$

$$\nabla f(-1, 0, 0) = D\mathbf{x} g_{\mathbf{x}}^{-1} \nabla \tilde{f}(\pi, 0)$$

$$= D\mathbf{x} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

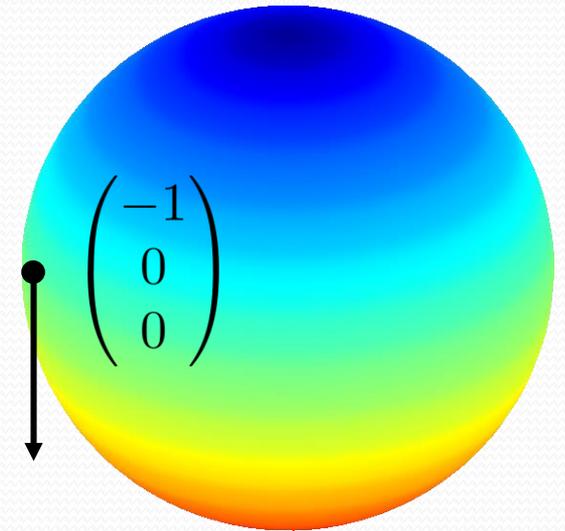
$$= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Second parametrization



$$D_{\mathbf{y}}(-1, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$$

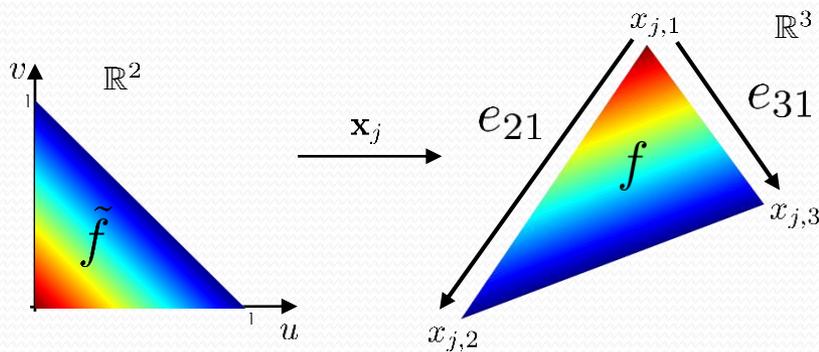
$$g_{\mathbf{y}}(-1, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (g_{\mathbf{y}}(-1, 0))^{-1}$$



$$\begin{aligned} \nabla f(-1, 0, 0) &= D_{\mathbf{y}} g_{\mathbf{y}}^{-1} \nabla \hat{f}(-1, 0) \\ &= D_{\mathbf{y}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \end{aligned}$$

Discretization: gradient

$$\mathbf{x}_j(u, v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



$$D\mathbf{x} = (\mathbf{x}_u, \mathbf{x}_v) = (e_{21}, e_{31})$$

$$g_j = \begin{pmatrix} \|e_{21}\|^2 & \langle e_{21}, e_{31} \rangle \\ \langle e_{21}, e_{31} \rangle & \|e_{31}\|^2 \end{pmatrix}$$

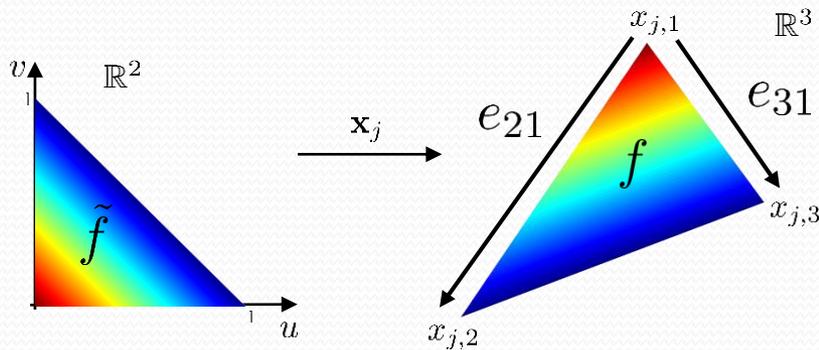
Recall that we are assuming functions to be linear within each triangle.

Our partial derivatives on the reference triangle can be simply approximated by the finite differences:

$$\frac{\partial \tilde{f}}{\partial u} = f(x_{j,2}) - f(x_{j,1}) \quad \frac{\partial \tilde{f}}{\partial v} = f(x_{j,3}) - f(x_{j,1})$$

Discretization: gradient

$$\mathbf{x}_j(u, v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$



$$D\mathbf{x} = (\mathbf{x}_u, \mathbf{x}_v) = (e_{21}, e_{31})$$

$$g_j = \begin{pmatrix} \|e_{21}\|^2 & \langle e_{21}, e_{31} \rangle \\ \langle e_{21}, e_{31} \rangle & \|e_{31}\|^2 \end{pmatrix}$$

The discrete gradient is then given by:

$$\nabla f = D\mathbf{x} g^{-1} \nabla \tilde{f} = (e_{21}, e_{31}) \begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix}^{-1} \begin{pmatrix} f(x_{j,2}) - f(x_{j,1}) \\ f(x_{j,3}) - f(x_{j,1}) \end{pmatrix}$$

Also observe that:
$$\begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix}^{-1} = \begin{pmatrix} G_j & -F_j \\ -F_j & E_j \end{pmatrix} \frac{1}{\det g_j}$$

Discretization: gradient norm

$$\|\nabla f\| = \sqrt{\nabla \tilde{f}^\top g^{-1} \nabla \tilde{f}}$$

We simply obtain:

$$\begin{aligned} \|\nabla f\| &= \sqrt{(f_u, f_v) \begin{pmatrix} G_j & -F_j \\ -F_j & E_j \end{pmatrix} \begin{pmatrix} f_u \\ f_v \end{pmatrix} \frac{1}{\sqrt{\det g_j}}} \\ &= \sqrt{\frac{f_u^2 G_j - 2f_u f_v F_j + f_v^2 E_j}{\det g_j}} \end{aligned}$$

Note that, since we take f to be linear, the gradient ∇f is constant within each triangle.

Discretization: total variation

The total variation of a function $f : S \rightarrow \mathbb{R}$ is defined as the integral of the norm of its gradient. For functions on manifolds, this corresponds to:

$$TV_S(f) = \sum_j \int_{T_j} \|\nabla f(x)\| da$$

We know how to compute integrals of functions on triangulated meshes. We simply get, for each triangle:

$$\begin{aligned} \int_{T_j} \|\nabla f(x)\| da &= \int_0^1 \int_0^{1-u} \sqrt{\frac{f_u^2 - 2f_u f_v F_j + f_v^2 E_j}{\det g_j}} \sqrt{\det g_j} dudv \\ &= \int_0^1 \int_0^{1-u} \sqrt{f_u^2 - 2f_u f_v F_j + f_v^2 E_j} dudv \\ &= \frac{1}{2} \sqrt{f_u^2 - 2f_u f_v F_j + f_v^2 E_j} \end{aligned}$$

The gradient

Demo Time!



Suggested reading

- *Differential geometry of curves and surfaces*. Do Carmo – Chapters 2.5, Appendix 2.B
- *Differential Geometry: Curves – Surfaces – Manifolds*. W. Kühnel – Chapter 3A