Multiple View Geometry: Solution Exercise Sheet 3

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Part I: Theory

1. (a) $M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (b) $M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (c) $M = \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (d) $M = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R & RT \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_1 t_x \\ r_{21} & r_{22} & r_{23} & r_2 t_y \\ r_{31} & r_{32} & r_{33} & r_3 t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

   where r_1, r_2, r_3 are the row vectors of R: $R = \begin{pmatrix} — r_1 — \\ — r_2 — \\ — r_3 — \end{pmatrix}$.

2. Let $M := (M_1 - M_2) =: \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$.

   "$\Rightarrow$":
   We show that M is skew-symmetric by distinguishing diagonal and off-diagonal elements of M:

   (a) $\forall i\colon 0 = e_i^T M e_i = m_{ii}$ where $e_i =$ i-th unit vector
   (b) $\forall i \neq j\colon 0 = (e_i + e_j)^T M (e_i + e_j)$ where $e_j =$ j-th unit vector
   $= m_{ii} + m_{jj} + m_{ij} + m_{ji} \Rightarrow m_{ij} = -m_{ij}$

   hence, $m_{ii} = 0$ and $m_{ij} = -m_{ij}$, i.e. M is skew-symmetric.

   "$\Leftarrow$":
   using $M = -M^T$, we directly calculate

$$\forall x\colon x^T M x = (x^T M x)^T = x^T M^T x = -(x^T M x)$$
$$\Rightarrow x^T M x = 0$$

3. We know: $\omega = (\omega_1\ \omega_2\ \omega_3)^T$ with $||\omega|| = 1$ and $\hat{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$.

(a)

$$
\begin{aligned}
\hat{\omega}^2 &= \begin{pmatrix} -(\omega_2^2+\omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2+\omega_3^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2+\omega_2^2) \end{pmatrix} \\
&= \begin{pmatrix} \omega_1^2 - \underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 - \underbrace{(\omega_2^2+\omega_1^2+\omega_3^2)}_{1} & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 - \underbrace{(\omega_3^2+\omega_1^2+\omega_2^2)}_{1} \end{pmatrix} \\
&= \begin{pmatrix} \omega_1^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & \omega_2^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & \omega_3^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \omega\omega^\top - \mathrm{I}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\omega}^3 &= \hat{\omega}\,\hat{\omega}^2 \\
&= \hat{\omega}\,(\omega\omega^T - I) \\
&= \hat{\omega}\,\omega\,(\omega^T) - \hat{\omega}I \\
&= (\omega\times\omega)\,\omega^T - \hat{\omega} \\
&= -\hat{\omega} \qquad\qquad (\text{as } \omega\times\omega = 0)
\end{aligned}
$$

Alternative solution for $\hat{\omega}^3$:

$$
\begin{aligned}
\hat{\omega}^3 &= \begin{pmatrix} -(\omega_2^2+\omega_3^2) & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -(\omega_1^2+\omega_2^2) & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -(\omega_1^2+\omega_2^2) \end{pmatrix} \cdot \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \omega_3\cdot\underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & -\omega_2\cdot\underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} \\ -\omega_3\cdot\underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & 0 & \omega_1\cdot\underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} \\ \omega_2\cdot\underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & -\omega_1\cdot\underbrace{(\omega_1^2+\omega_2^2+\omega_3^2)}_{1} & 0 \end{pmatrix} \\
&= -\hat{\omega}
\end{aligned}
$$

(b) The formulas for n even and odd can be found by writing down the solutions for $n = 1, \dots, 6$:

$$\begin{aligned}
&\hat{\omega} \\
&\hat{\omega}^2 \\
\hat{\omega}^3 &= -\hat{\omega} \\
\hat{\omega}^4 &= -\hat{\omega}^2 && \text{as: } \hat{\omega}^4 = \hat{\omega}^3\hat{\omega} = -\hat{\omega}\hat{\omega} = -\hat{\omega}^2 \\
\hat{\omega}^5 &= \hat{\omega} && \text{as: } \hat{\omega}^5 = \hat{\omega}^4\hat{\omega} = -\hat{\omega}^2\hat{\omega} = -\hat{\omega}^3 = -(-\hat{\omega}) = \hat{\omega} \\
\hat{\omega}^6 &= \hat{\omega}^2 && \text{as: } \hat{\omega}^6 = \hat{\omega}^5\hat{\omega} = \hat{\omega}\hat{\omega} = \hat{\omega}^2
\end{aligned}$$

For even numbers:

$$\begin{aligned}
&\hat{\omega}^2 \\
\hat{\omega}^4 &= -\hat{\omega}^2 \\
\hat{\omega}^6 &= \hat{\omega}^2
\end{aligned}$$

For odd numbers:

$$\begin{aligned}
&\hat{\omega} \\
\hat{\omega}^3 &= -\hat{\omega} \\
\hat{\omega}^5 &= \hat{\omega}
\end{aligned}$$

$$\begin{aligned}
n \text{ even:} \quad \hat{\omega}^n &= (-1)^{\frac{n}{2}+1}\, \hat{\omega}^2 \\
n \text{ odd:} \quad \hat{\omega}^n &= (-1)^{\frac{n-1}{2}}\, \hat{\omega}
\end{aligned}$$

Proof via complete induction:

i. For even numbers n:
   - $n = 2: \quad \hat{\omega}^2 = (-1)^{\frac{2}{2}+1}\hat{\omega}^2$
   - Induction step $n \to n+2$:

$$\begin{aligned}
\hat{\omega}^{n+2} &= \hat{\omega}^n \cdot \hat{\omega}^2 \\
&= (-1)^{\frac{n}{2}+1} \cdot \hat{\omega}^2 \cdot \hat{\omega}^2 && \text{(assumption)} \\
&= (-1)^{\frac{n}{2}+1} \cdot \hat{\omega}^3 \cdot \hat{\omega} \\
&\overset{(a)}{=} (-1)^{\frac{(n+2)}{2}+1} \cdot \hat{\omega}^2
\end{aligned}$$

ii. For odd numbers n:

- $n = 3: \quad \hat{\omega}^3 = -\hat{\omega} = (-1)^{\frac{3-1}{2}} \hat{\omega}$
- Induction step $n \to n+2$:

$$\begin{aligned}
\hat{\omega}^{n+2} &= \hat{\omega}^n \cdot \hat{\omega}^2 \\
&= (-1)^{\frac{n-1}{2}} \cdot \hat{\omega} \cdot \hat{\omega}^2 && \text{(assumption)} \\
&= (-1)^{\frac{n-1}{2}} \cdot \hat{\omega}^3 \\
&\overset{(a)}{=} (-1)^{\frac{n-1}{2}+1} \cdot \hat{\omega} \\
&= (-1)^{\frac{(n+2)-1}{2}} \cdot \hat{\omega}
\end{aligned}$$

(c) We know: $\omega \in \mathbb{R}^3$. Let $v = \frac{\hat{\omega}}{\|\omega\|}$ and $t = \|\omega\|$. Hence, $w = vt$.

$$\begin{aligned}
e^{\hat{\omega}} &= e^{\hat{\nu} t} \\
&= \sum_{n=0}^{\infty} \frac{(\hat{\nu} t)^n}{n!} \\
&\overset{(b)}{=} I + \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{t^{2n}}{(2n)!}}_{1-\cos(t)} \hat{\nu}^2 + \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}}_{\sin(t)} \hat{\nu} \\
&\overset{\text{(def.)}}{=} I + \frac{\hat{\omega}^2}{\|\omega\|^2} (1 - \cos(\|\omega\|)) + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|)
\end{aligned}$$