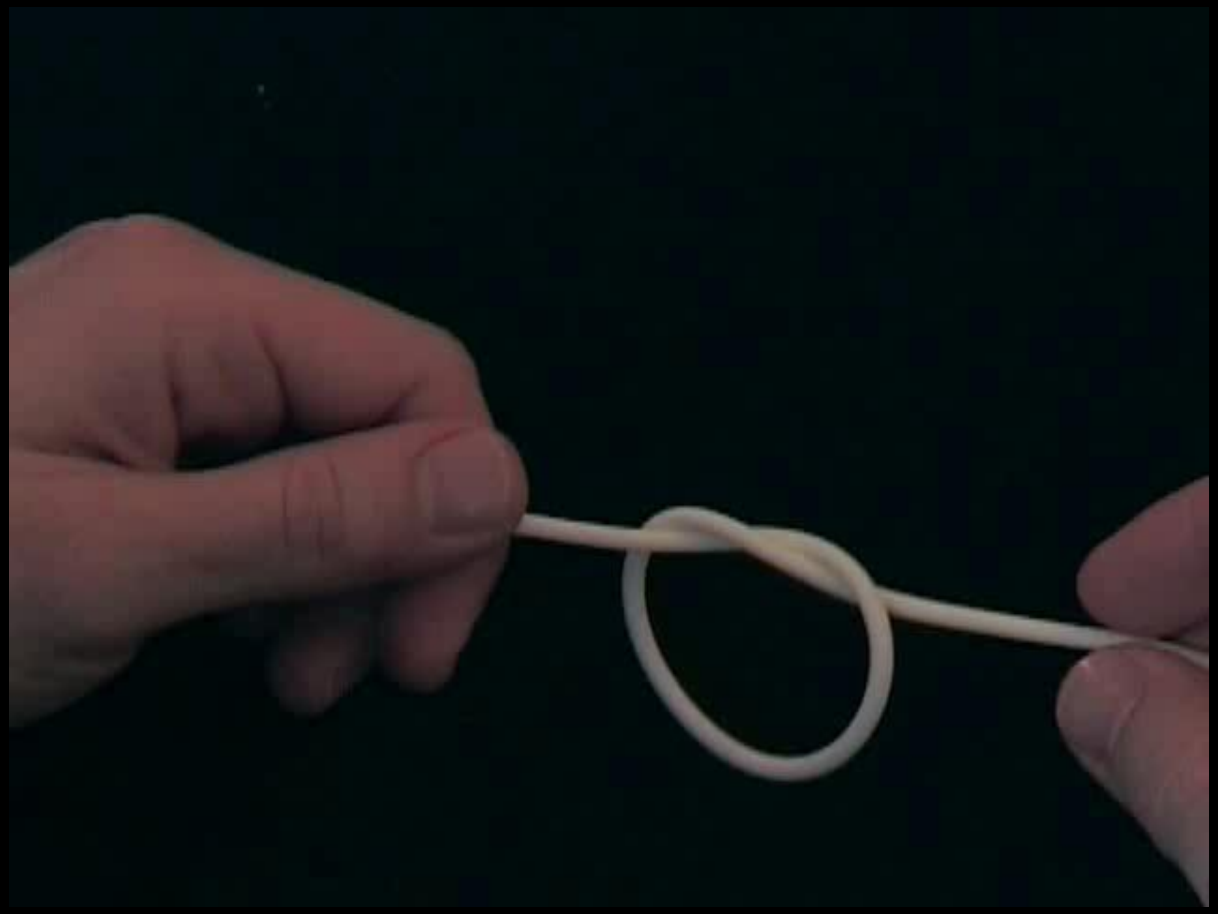
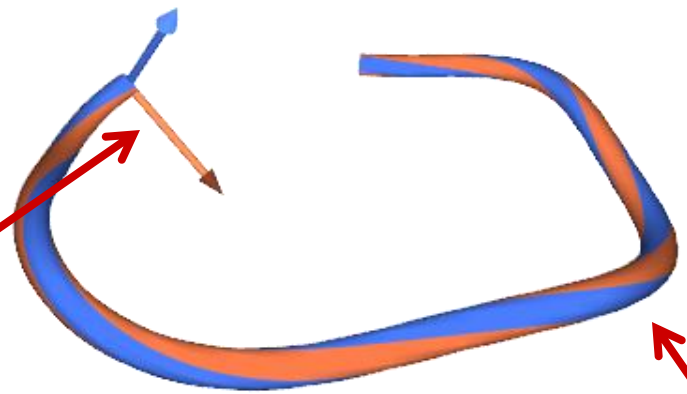
Discrete Elastic Rods

M. Bergou, M. Wardetzky, S. Robinson, B. Audoly, E. Grinspun,
ACM Transactions on Graphics (SIGGRAPH) 2008



Kirchhoff Rod Model

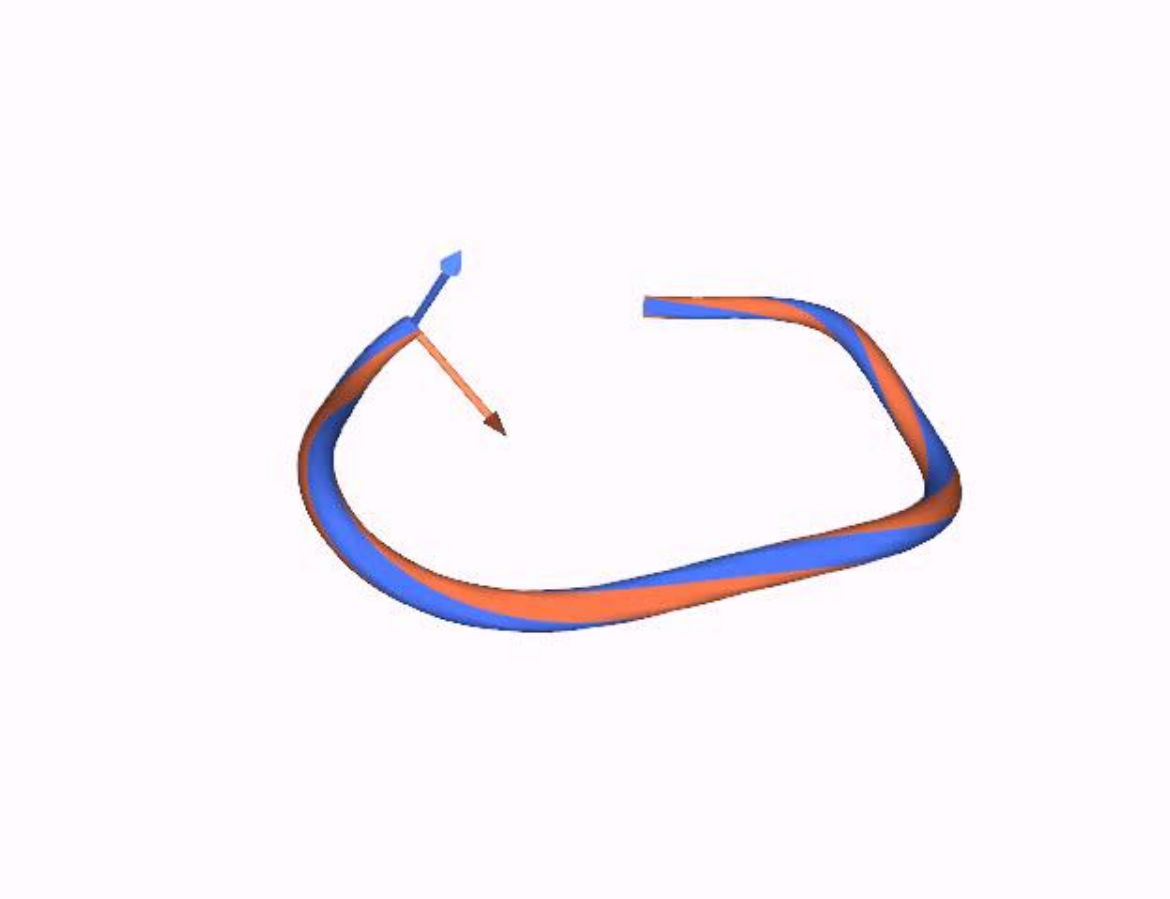


adapted frame

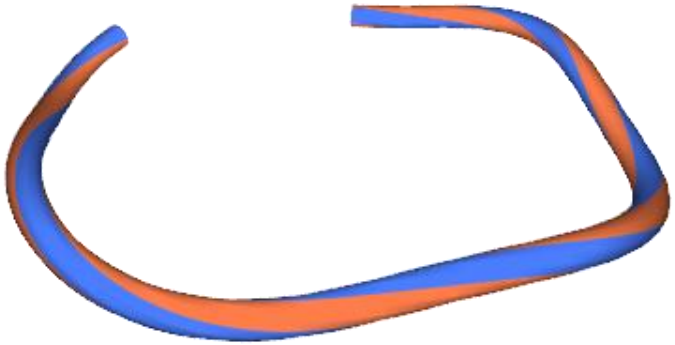
$\{\mathbf{m}_1, \mathbf{m}_2, \boxed{\mathbf{t} = \boldsymbol{\gamma}'(s)}\}$

curve $\boldsymbol{\gamma}(s)$

Kirchhoff Rod Model

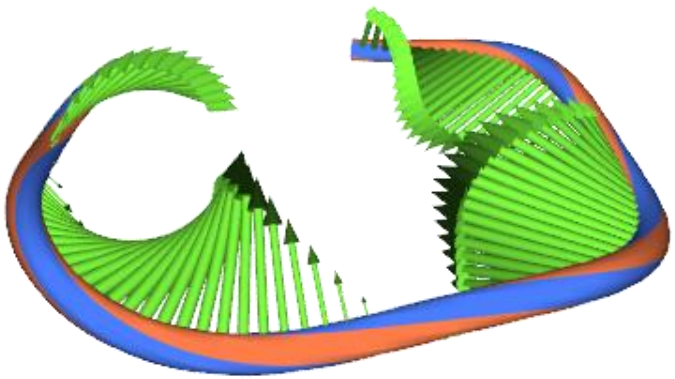


Energy of an Elastic Rod



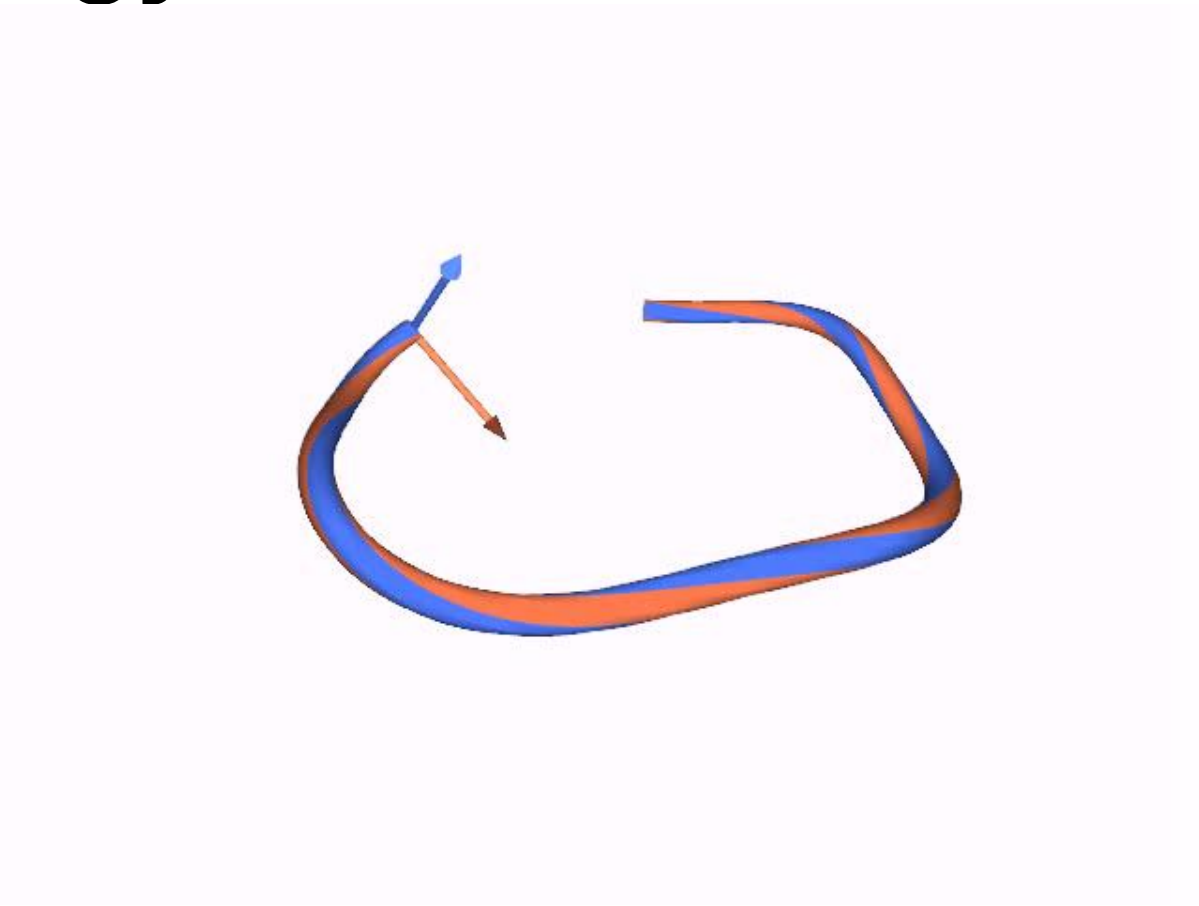
Energy of an Elastic Rod

curvature
$\kappa = \gamma''$



Energy of an Elastic Rod

curvature
$\kappa = \gamma''$

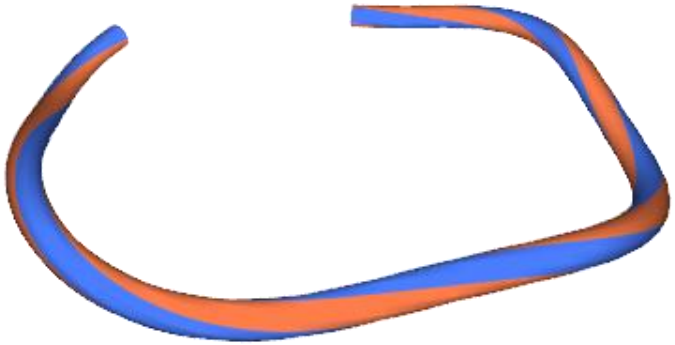


twist
$\mathbf{m}_1{}' \cdot \mathbf{m}_2$

Energy of an Elastic Rod

curvature

$\boldsymbol{\kappa} = \boldsymbol{\gamma}''$



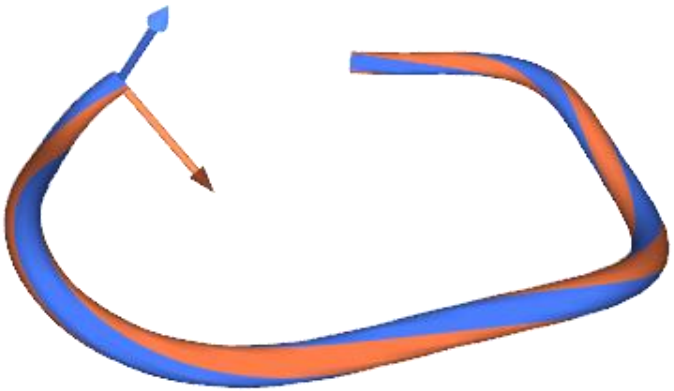
twist

$\mathbf{m}_1{}' \cdot \mathbf{m}_2$

$$\frac{1}{2}\int \alpha \boldsymbol{\kappa}^2 \mathrm{d}s + \frac{1}{2}\int \beta m^2 \mathrm{d}s$$

bending twisting

Degrees of Freedom



Represent configuration

- centerline
- adapted material frame

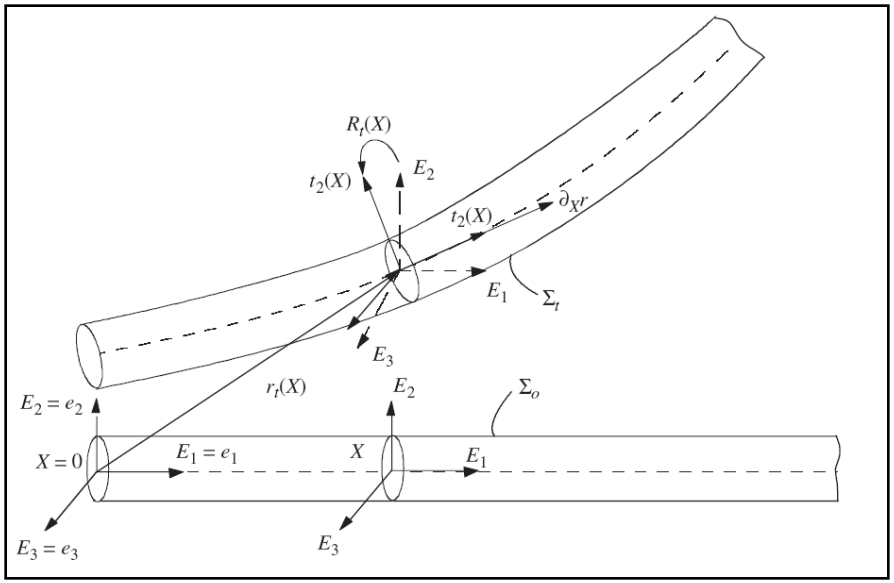
Everything else follows

- energy
- forces
- time stepping

Foundation

Curve-angle representation

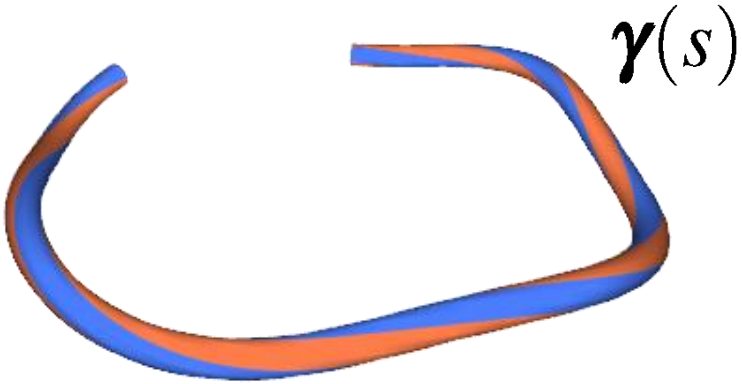
- [Langer & Singer ’96] [Boyer & Primault ’04], [Theetten *et al*. ’07]
- explicit centerline
- reference frame
- rotation angle



[Boyer & Primault ’04]

Curve + Angle Representation

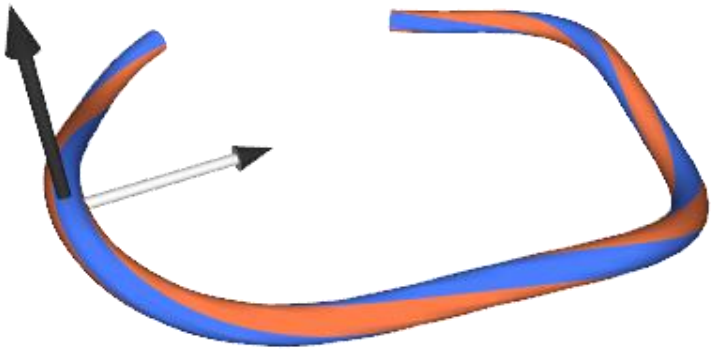
Explicit centerline



Curve + Angle Representation

Explicit centerline

Reference frame

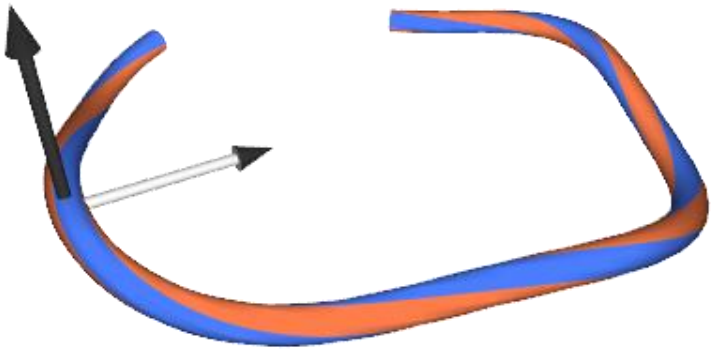


Curve + Angle Representation

Explicit centerline

Reference frame

- $\{\mathbf{u}, \mathbf{v}\}$ twist-free



Curve + Angle Representation

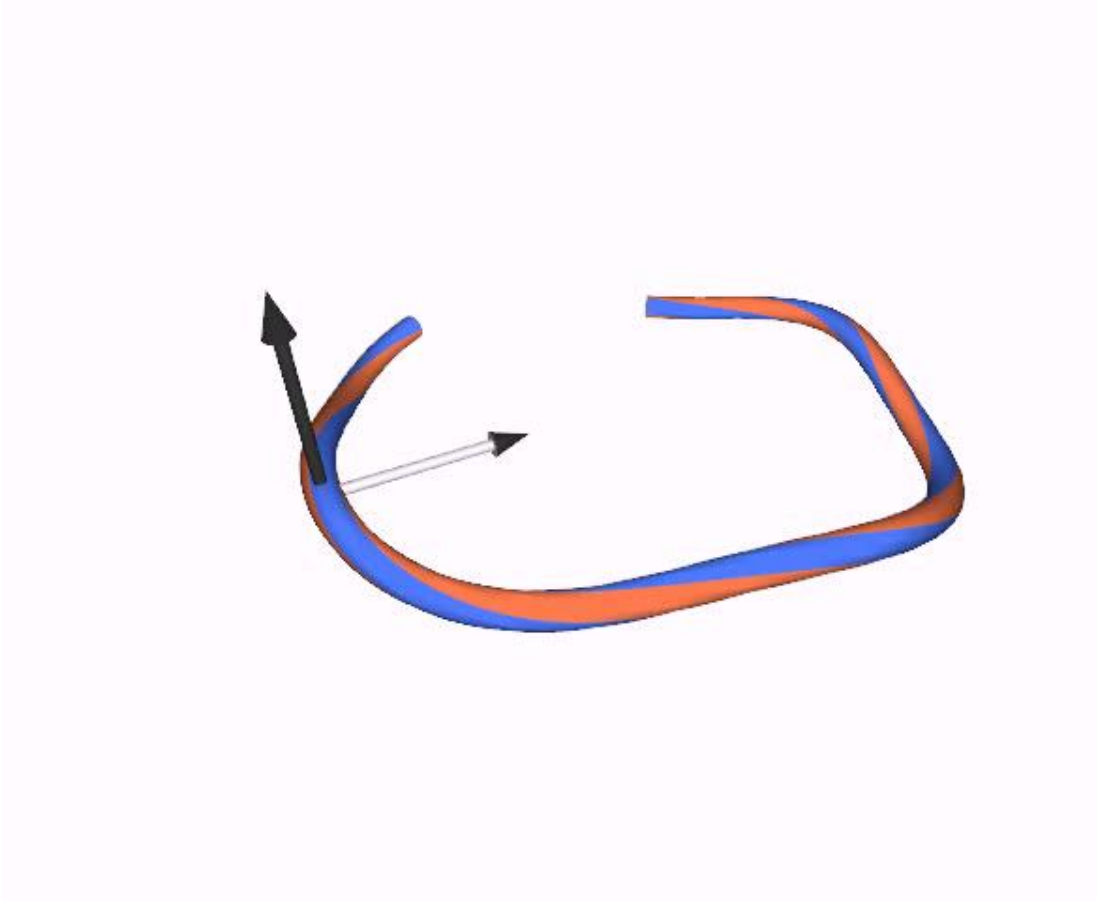
Explicit centerline

Reference frame

- $\{\mathbf{u}, \mathbf{v}\}$ twist-free

Material frame

- relative angle



Curve + Angle Representation

Explicit centerline

Reference frame

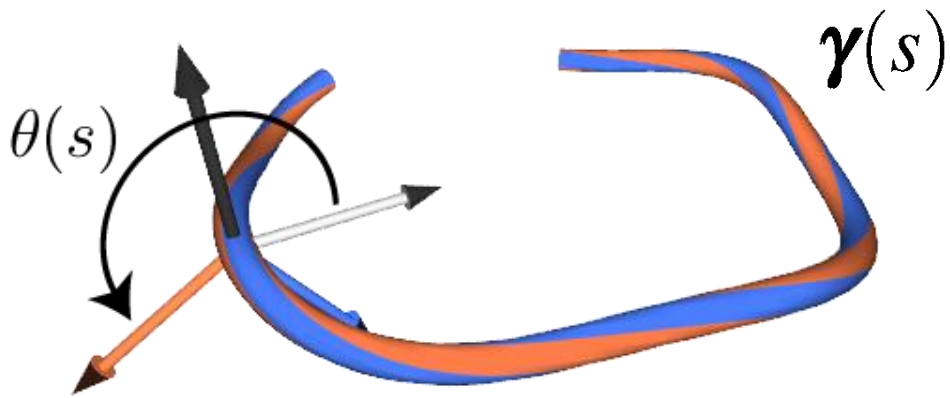
- $\{\mathbf{u}, \mathbf{v}\}$ twist-free

Material frame

- relative angle

Coordinates

- $\boldsymbol{\gamma}(s) : \mathbb{R} \rightarrow \mathbb{R}^3$
- $\boldsymbol{\theta}(s) : \mathbb{R} \rightarrow \mathbb{R}$



Curve + Angle Representation

Explicit centerline

Reference frame

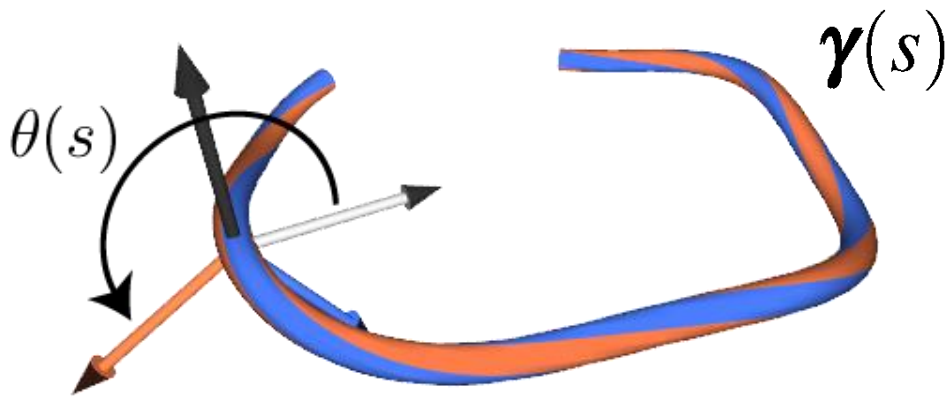
- $\{\mathbf{u}, \mathbf{v}\}$ twist-free

Material frame

- relative angle

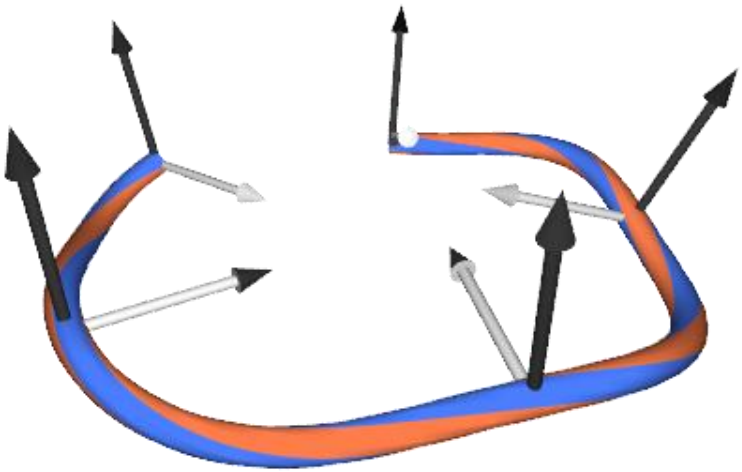
Coordinates

- $\boldsymbol{\gamma}(s) : \mathbb{R} \rightarrow \mathbb{R}^3$
- $\boldsymbol{\theta}(s) : \mathbb{R} \rightarrow \mathbb{R}$



Twist-Free Frame

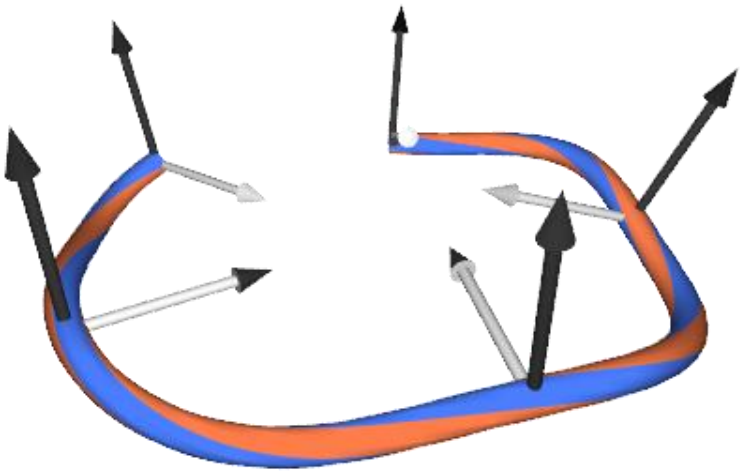
Twist-Free Frame



Requirements

1. no twist about tangent
2. frame must stay adapted

Parallel Transport



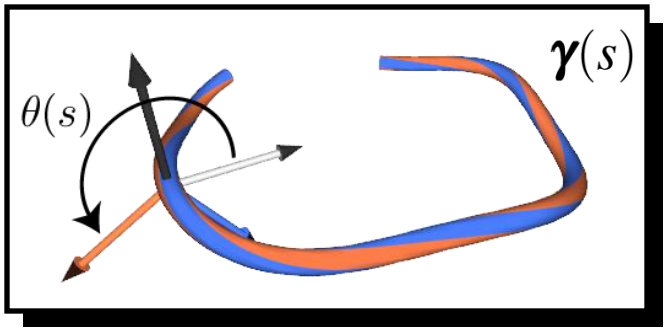
Requirements

1. no twist about tangent
2. frame must stay adapted

Rotation about binormal

$$\mathbf{u}' = \kappa \mathbf{b} \times \mathbf{u}$$
$$\mathbf{v}' = \kappa \mathbf{b} \times \mathbf{v}$$

Continuous Setting

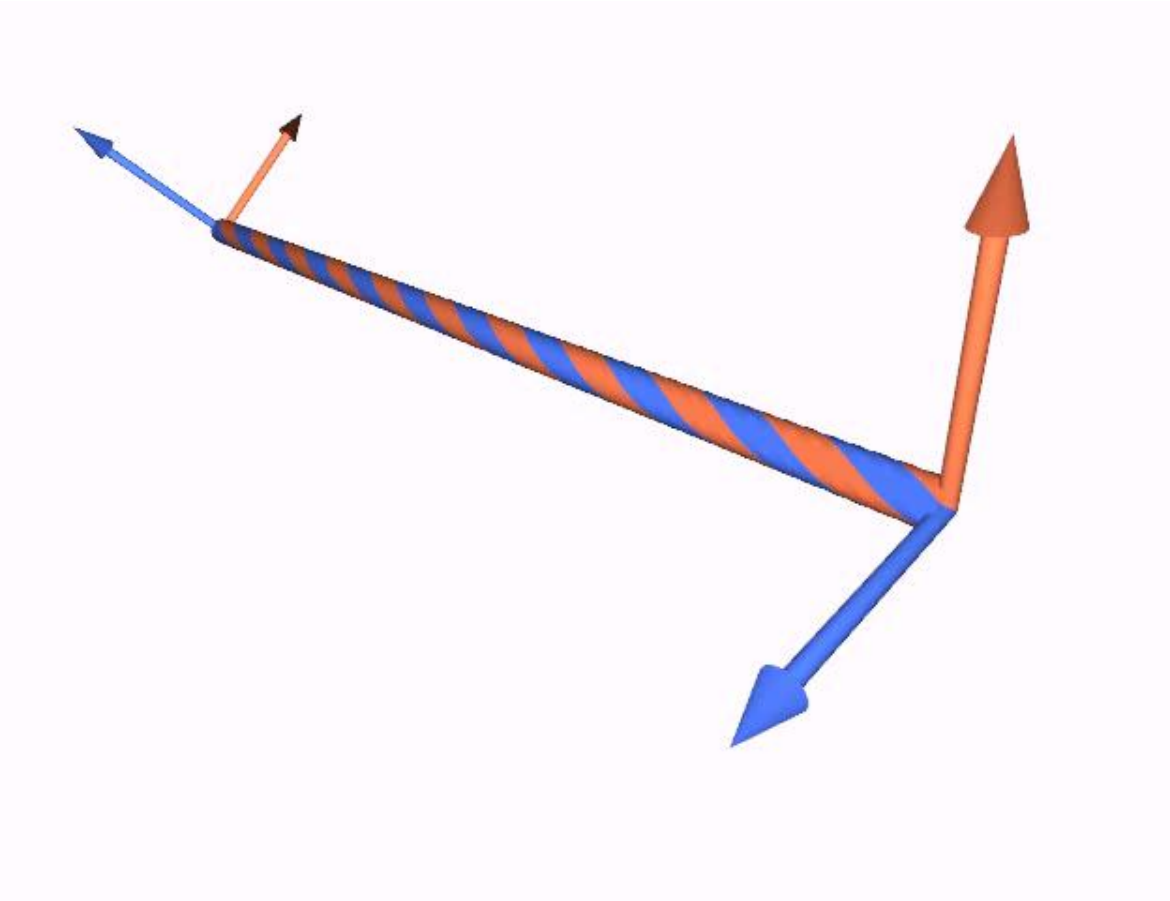


bending twisting

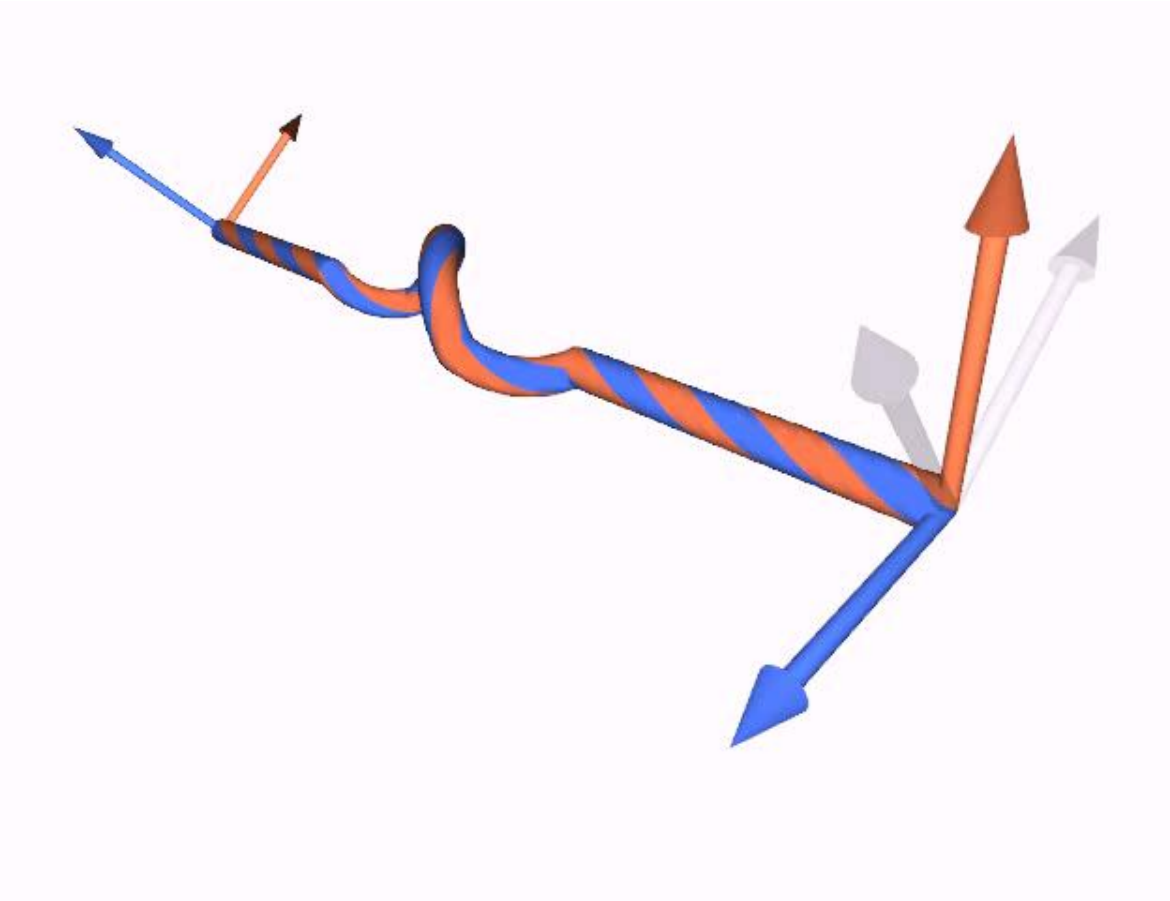
$$\frac{1}{2}\int \alpha \boldsymbol{\kappa}^2 \mathrm{d}s + \frac{1}{2}\int \beta m^2 \mathrm{d}s$$

$$\boldsymbol{\kappa} = \boxed{\boldsymbol{\gamma}''} \qquad m = \boxed{\theta'}$$

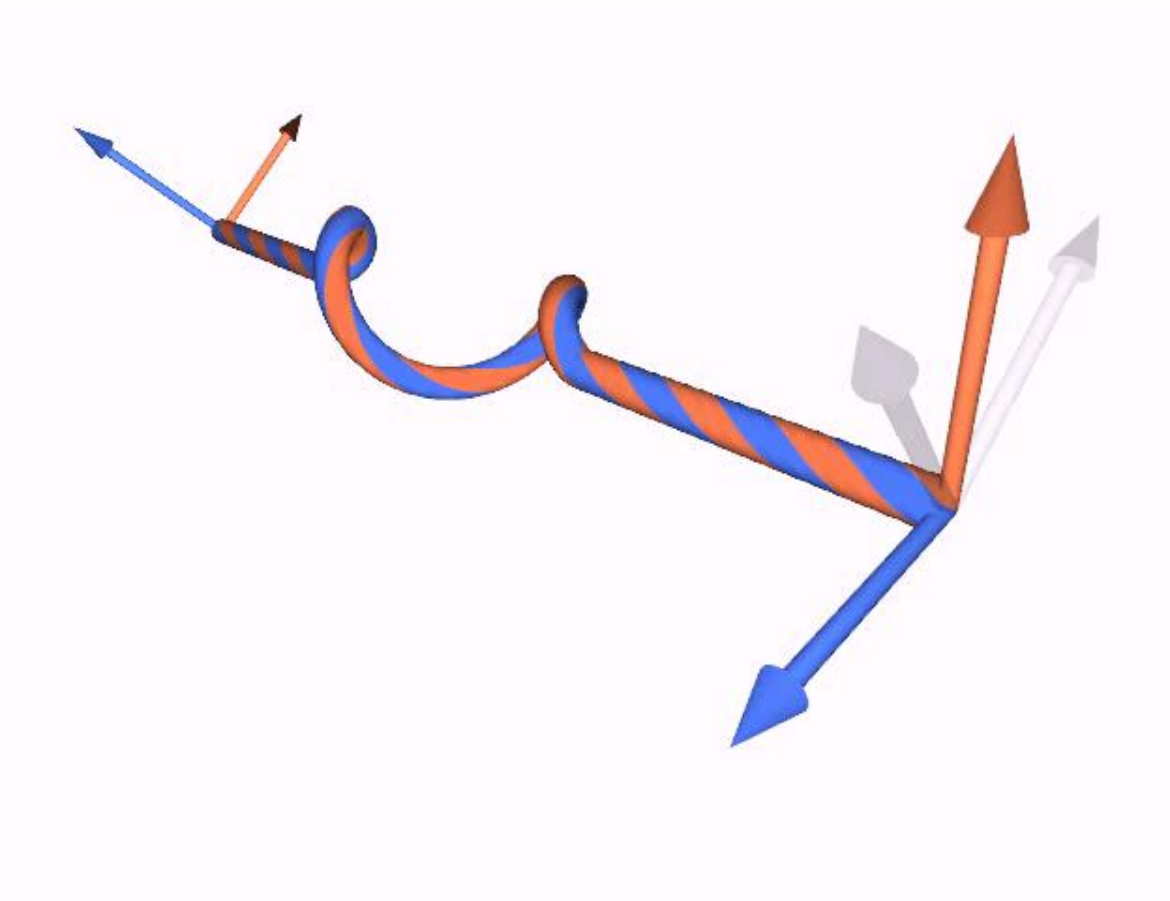
Bend & Twist Interaction



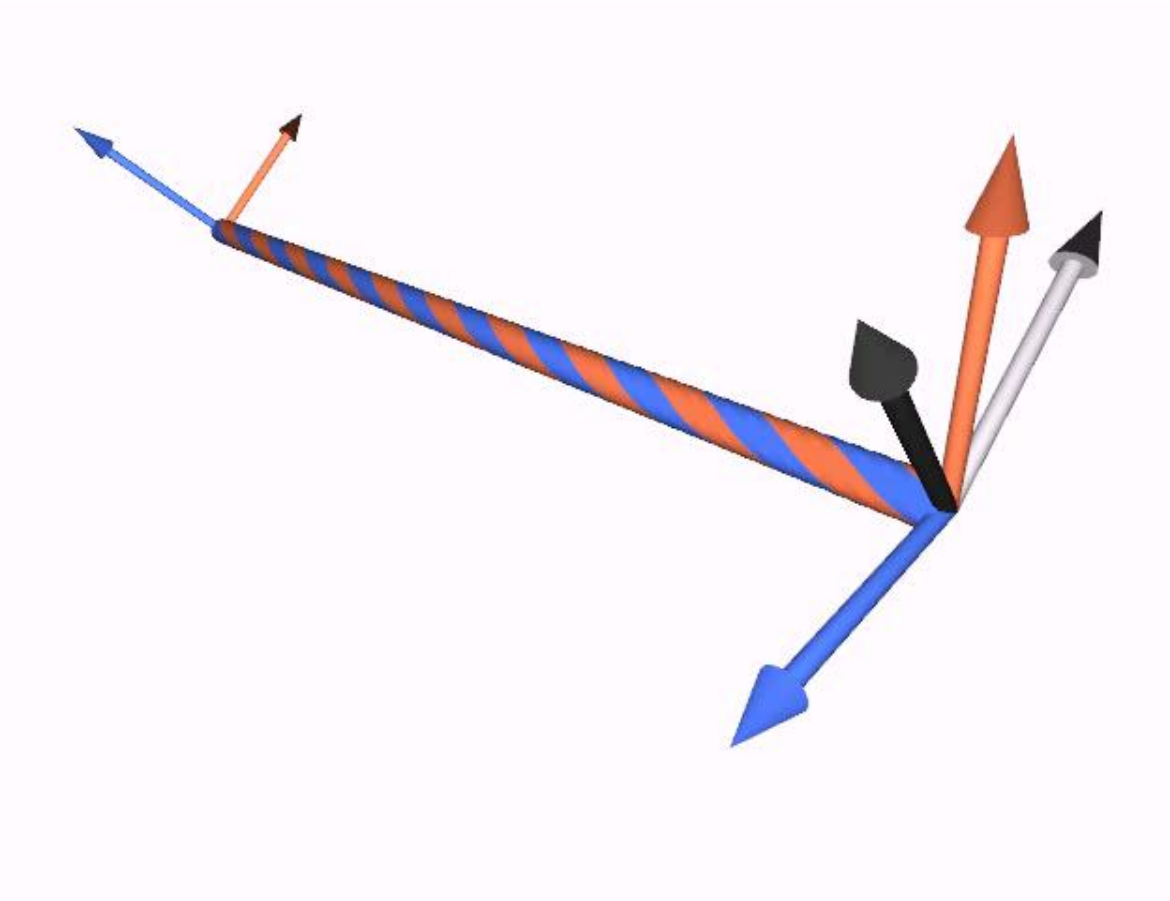
Bend & Twist Interaction



Bend & Twist Interaction



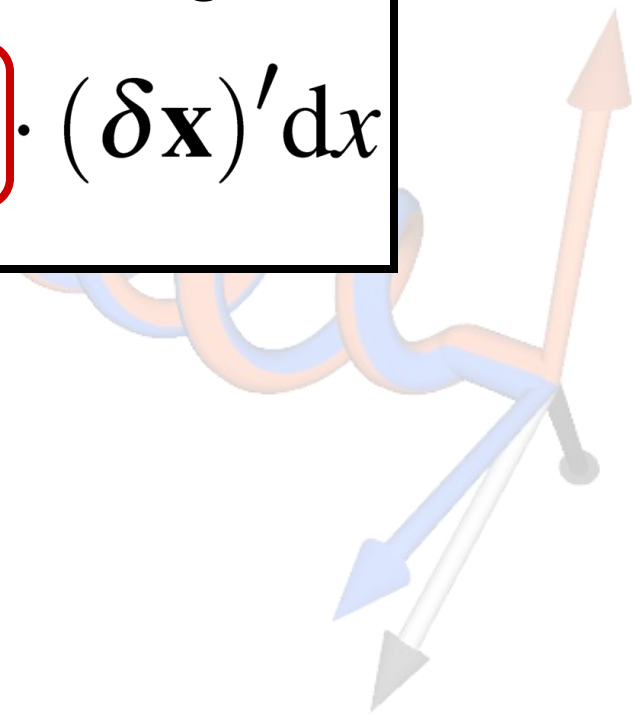
Bend & Twist Interaction



Bend & Twist Interaction

change in angle

$$-\int \boxed{(\boldsymbol{\kappa}\mathbf{b})} \cdot (\delta\mathbf{x})' \mathrm{d}x$$



Bend & Twist Interaction

$$\frac{1}{2}\int \alpha \boldsymbol{\kappa}^2 \mathrm{d}s + \frac{1}{2}\int \beta (\theta')^2 \mathrm{d}s$$

How do bending and twisting interact?

Both terms affect centerline

- bending force

  moves centerline toward straighter curve

- twisting force

  aligns twist-free & material frames

Transition to Discrete Setting

Build model from discrete building blocks

- discrete curve + angle representation
- parallel transport
- twist
- curvature

Curve + Angle Representation

Explicit centerline

Reference frame

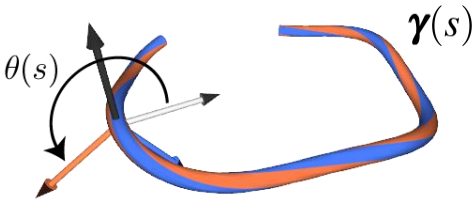
- $\{\mathbf{u}, \mathbf{v}\}$ twist-free

Material frame

- relative angle

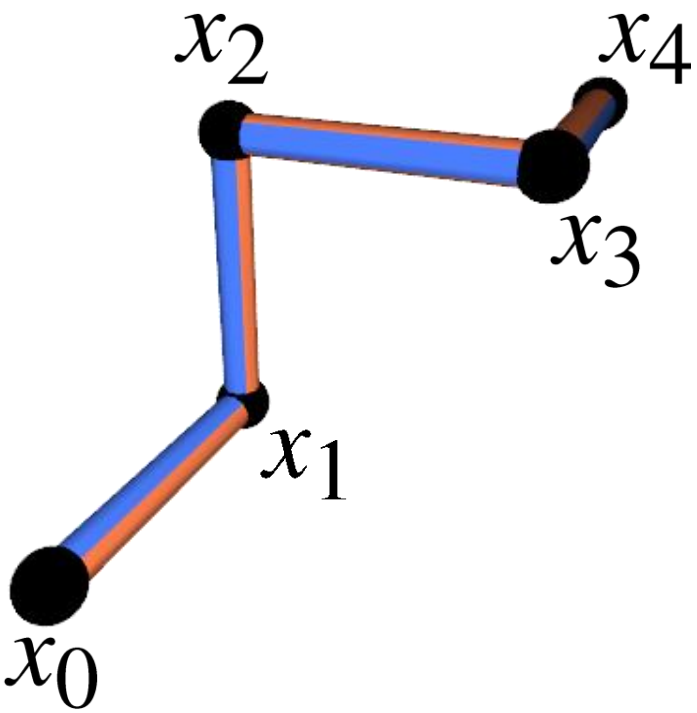
Coordinates

- $\boldsymbol{\gamma}(s) : \mathbb{R} \to \mathbb{R}^3$
- $\theta(s) : \mathbb{R} \to \mathbb{R}$



Discrete Curve + Angle

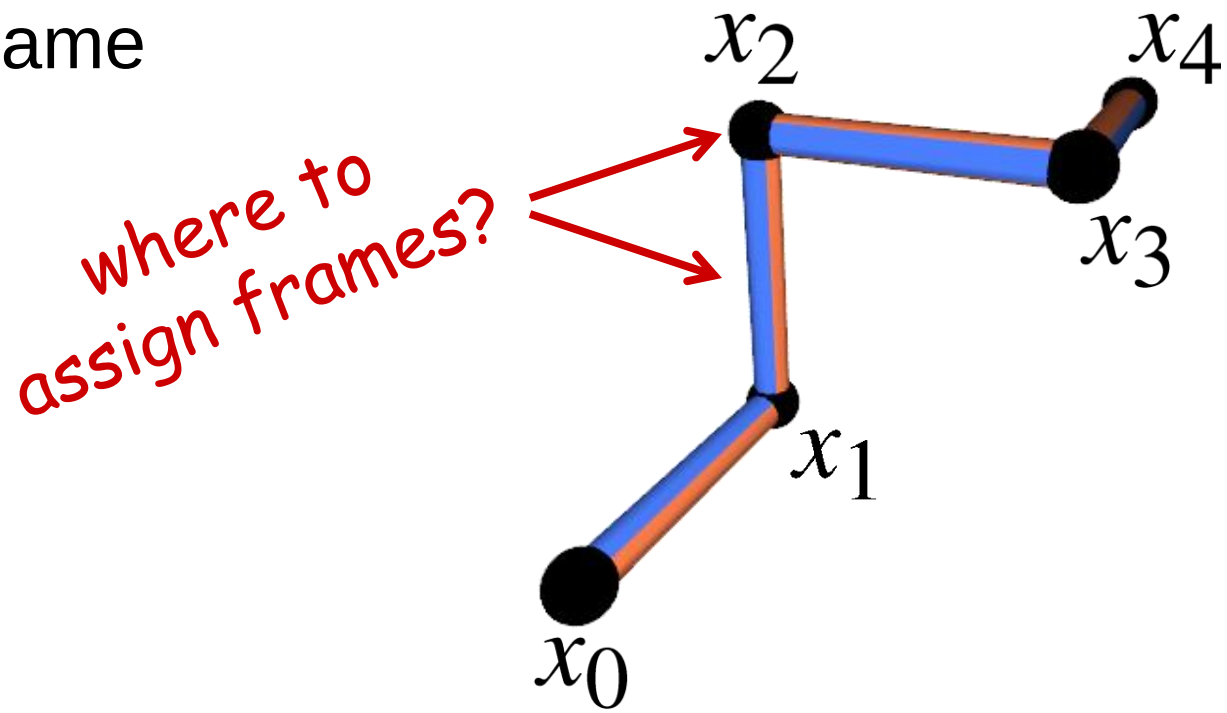
Explicit centerline



Discrete Curve + Angle

Explicit centerline

Reference frame

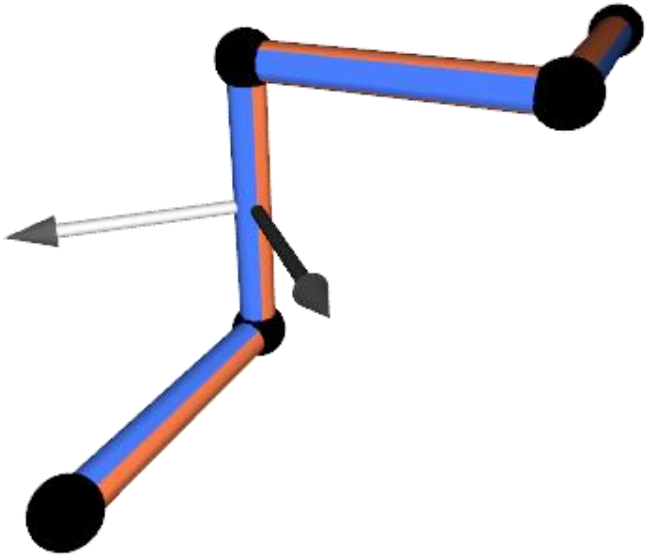


Discrete Curve + Angle

Explicit centerline

Reference frame

- $\{\mathbf{u}, \mathbf{v}\}$ twist-free



Discrete Curve + Angle

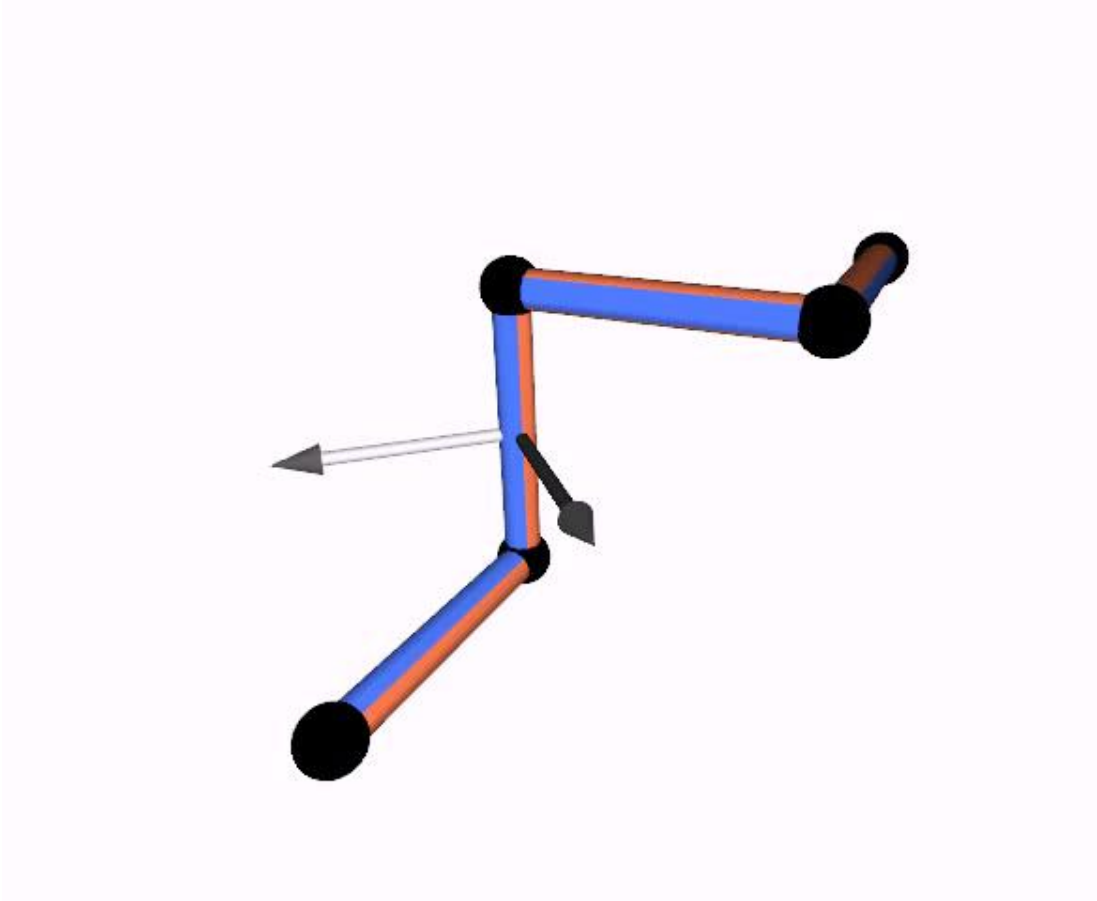
Explicit centerline

Reference frame

- $\{\mathbf{u}, \mathbf{v}\}$ twist-free

Material frame

- relative angle



Discrete Curve + Angle

Explicit centerline

Reference frame

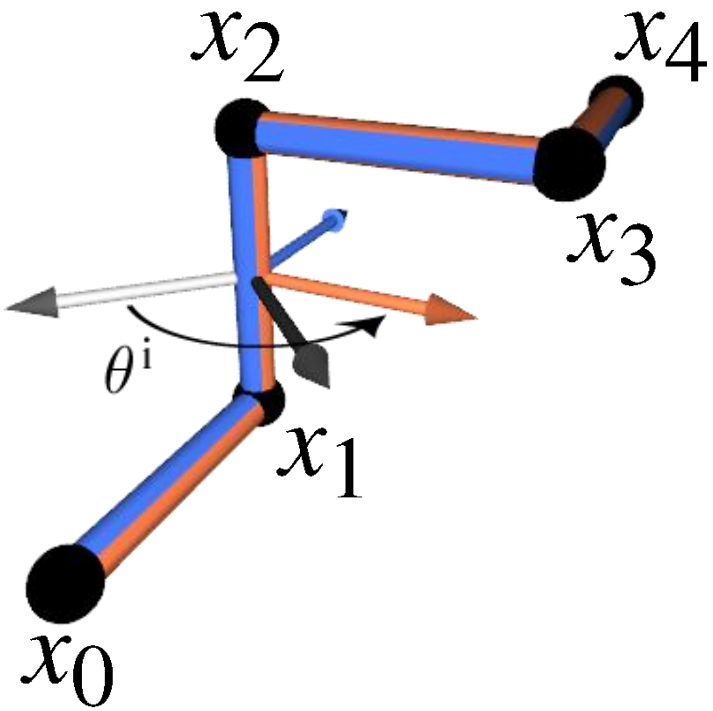
- $\{\mathbf{u}, \mathbf{v}\}$ twist-free

Material frame

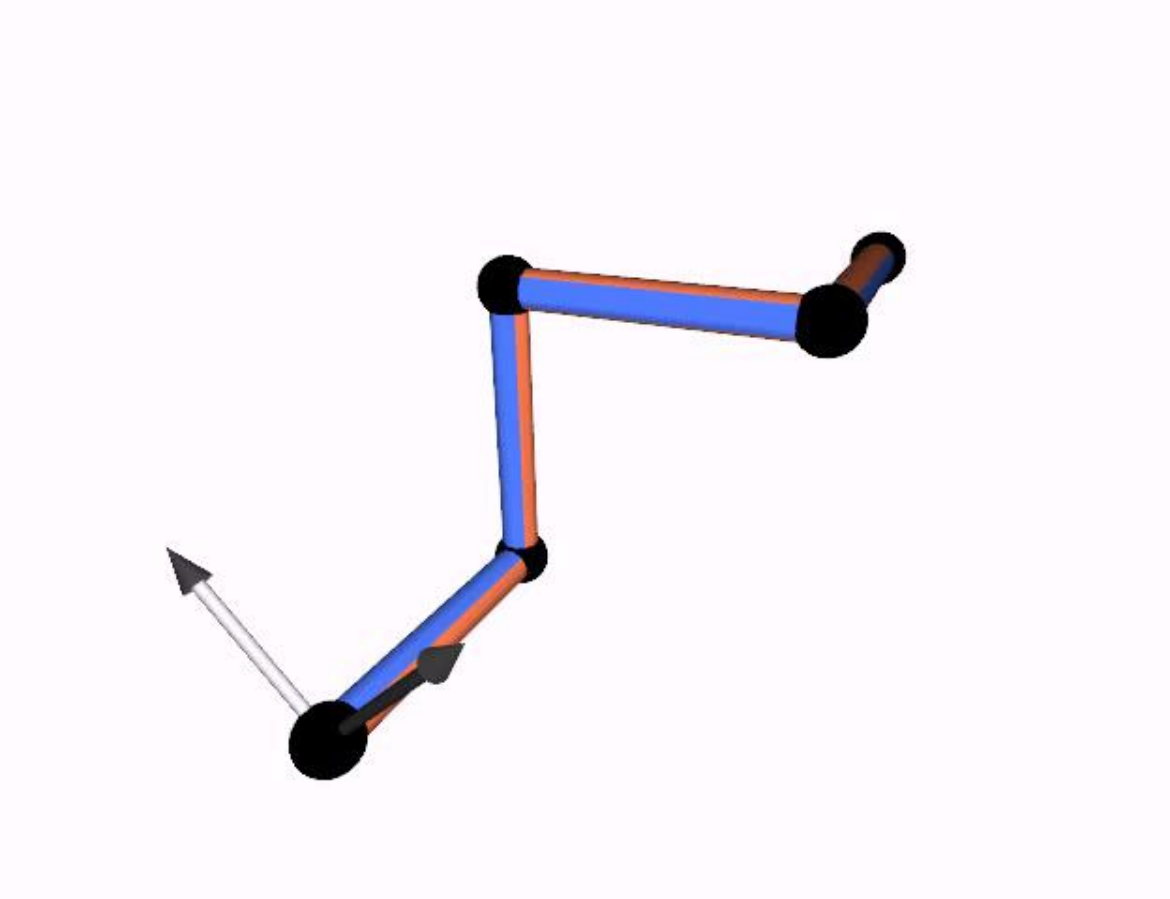
- relative angle

Degrees of freedom

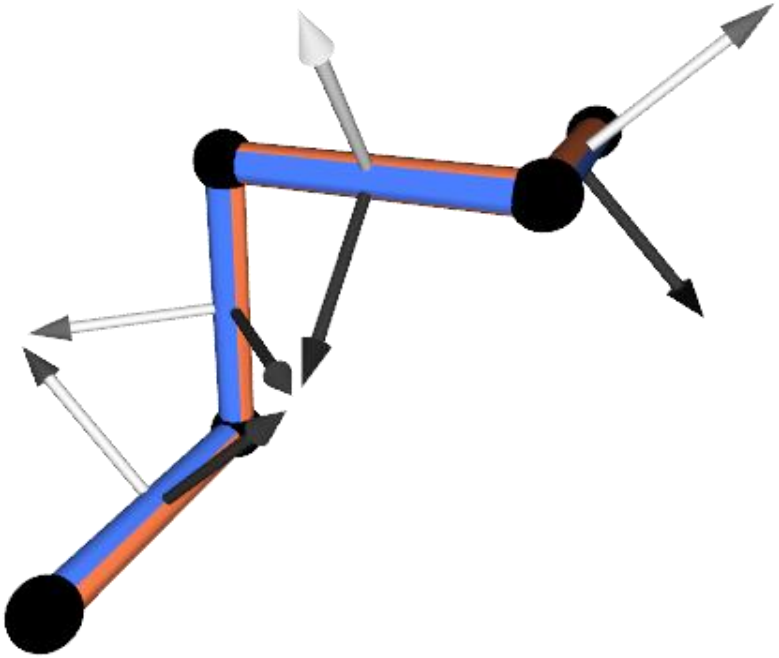
- $\{\mathbf{x}_0, \cdots, \mathbf{x}_{n+1}\} \in \mathbb{R}^{3(n+2)}$
- $\{\theta^0, \cdots, \theta^n\} \in \mathbb{R}^{n+1}$



Twist-Free Frame



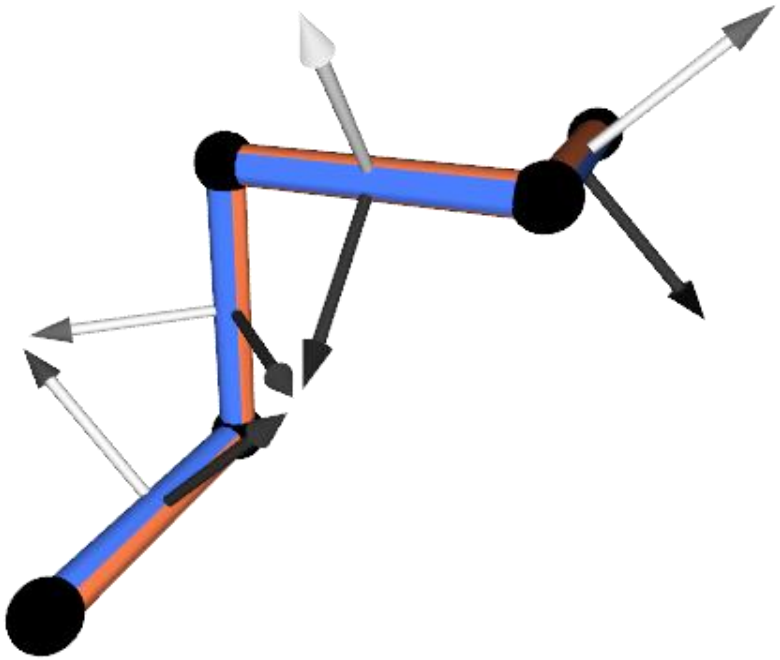
Twist-Free Frame



Requirements

1. no twist about tangent
2. frame must stay adapted

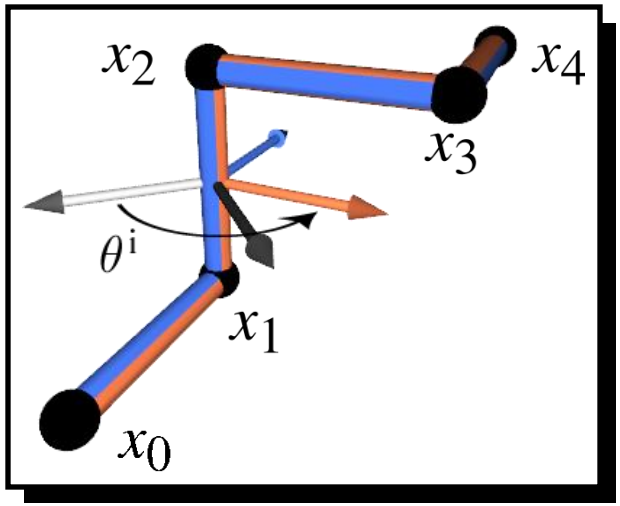
Discrete Parallel Transport



Requirements
1. no twist about tangent
2. frame must stay adapted

Translate along edge
Rotate at vertex
about binormal

Discrete Energy

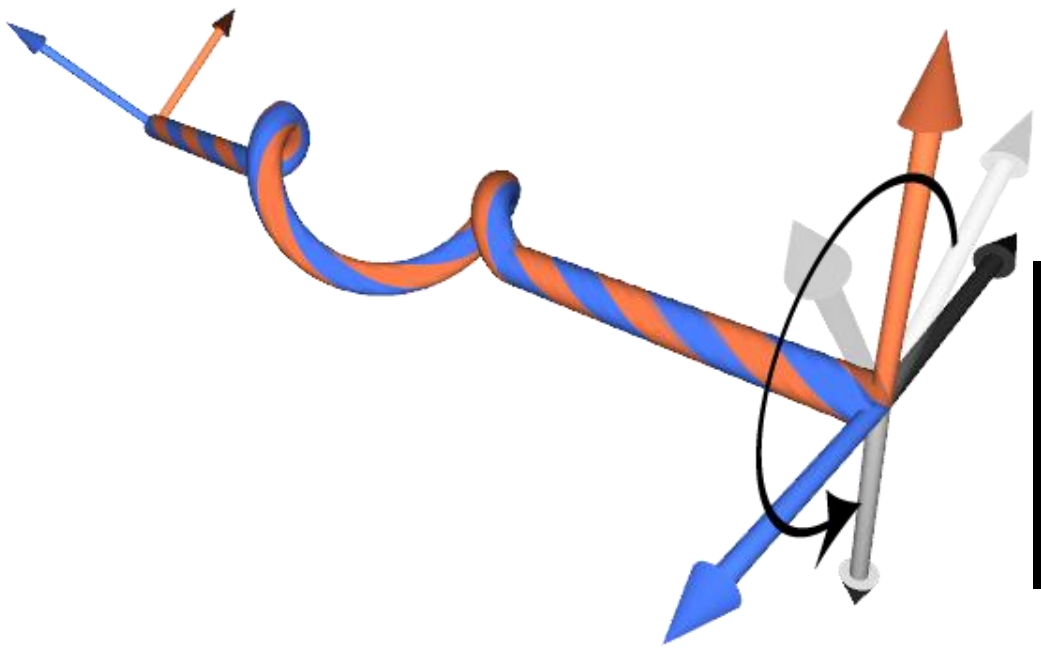


bending twisting

$$\frac{1}{2}\int \alpha \boldsymbol{\kappa}^2 \mathrm{d}s + \frac{1}{2}\int \beta (\theta')^2 \mathrm{d}s$$

$$\boldsymbol{\kappa}_i = ? \qquad \theta_i' = \frac{\theta^i - \theta^{i-1}}{l_i}$$

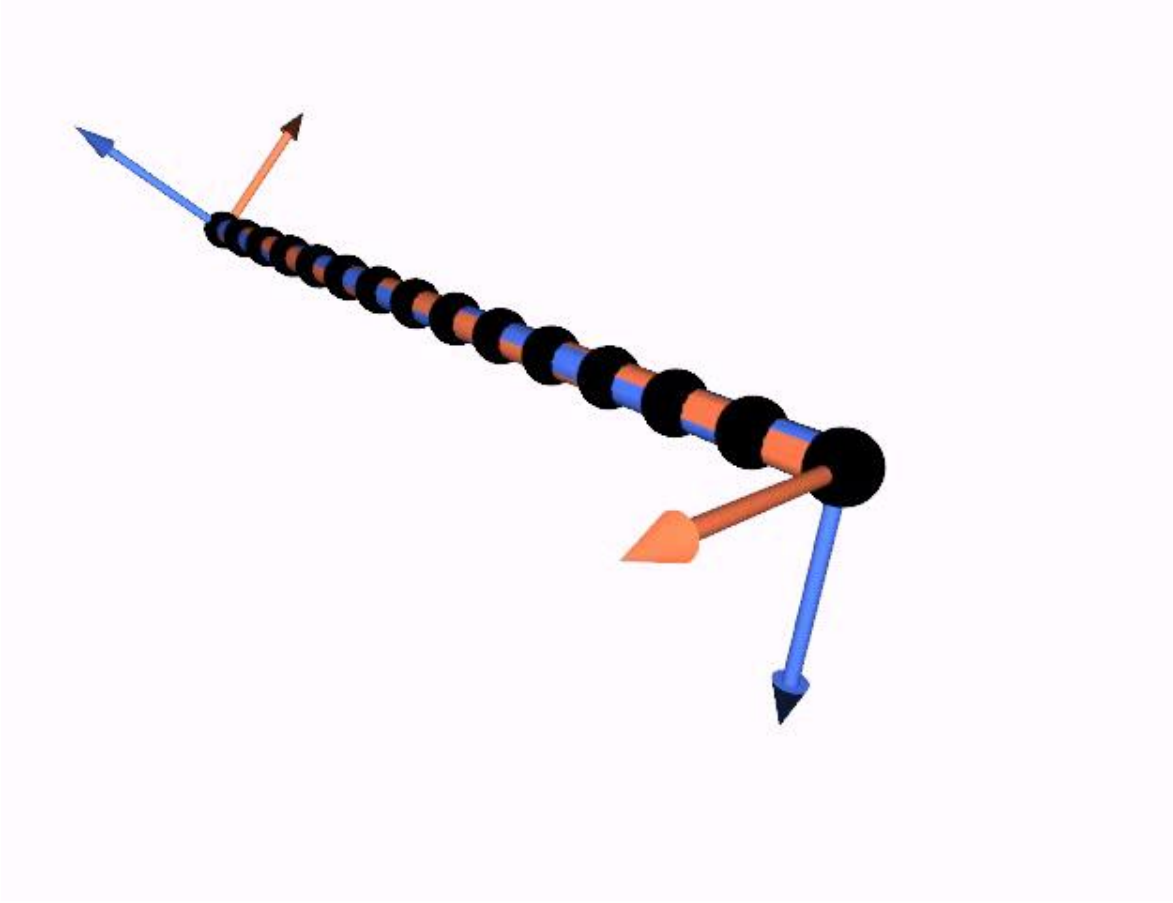
Curvature



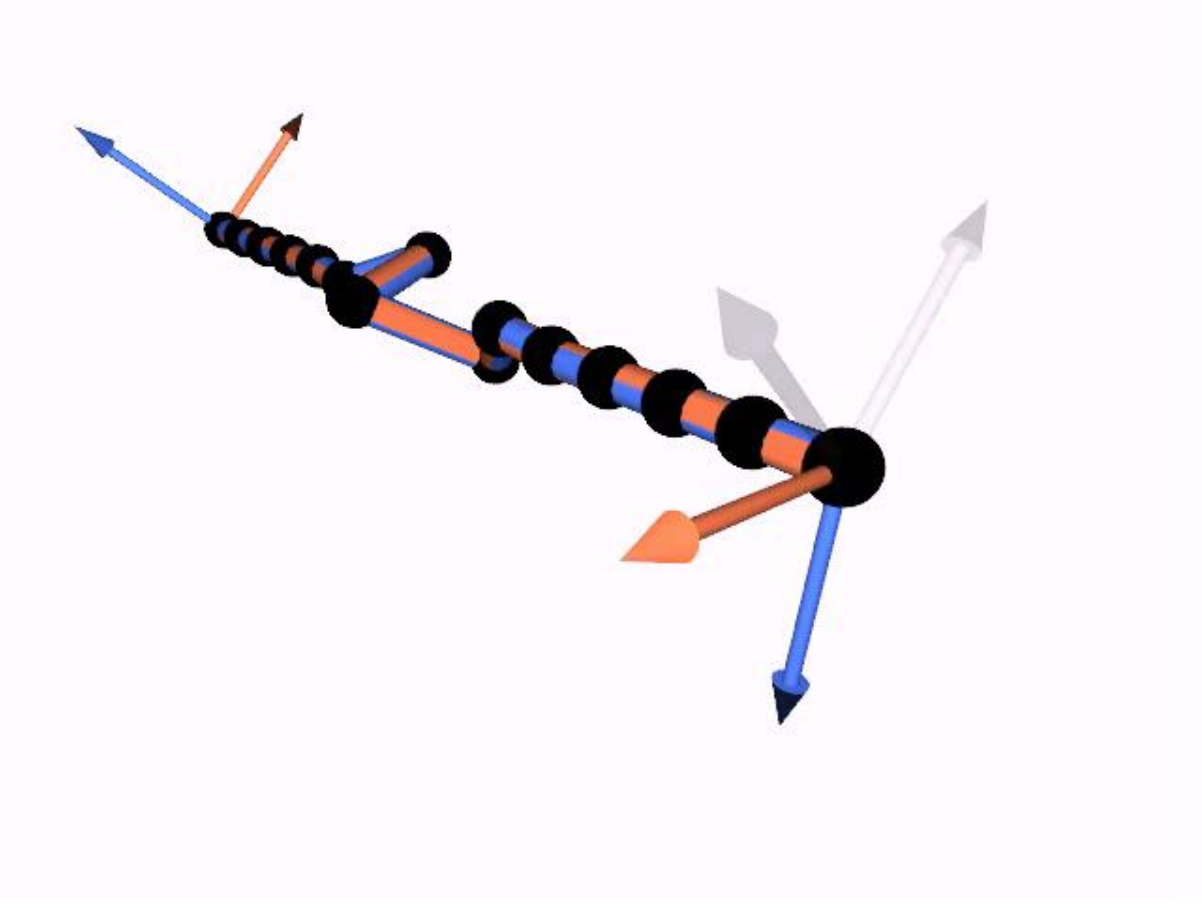
change in angle

$$-\int \boxed{(\kappa \mathbf{b})} \cdot (\delta \mathbf{x})' \mathrm{d}s$$

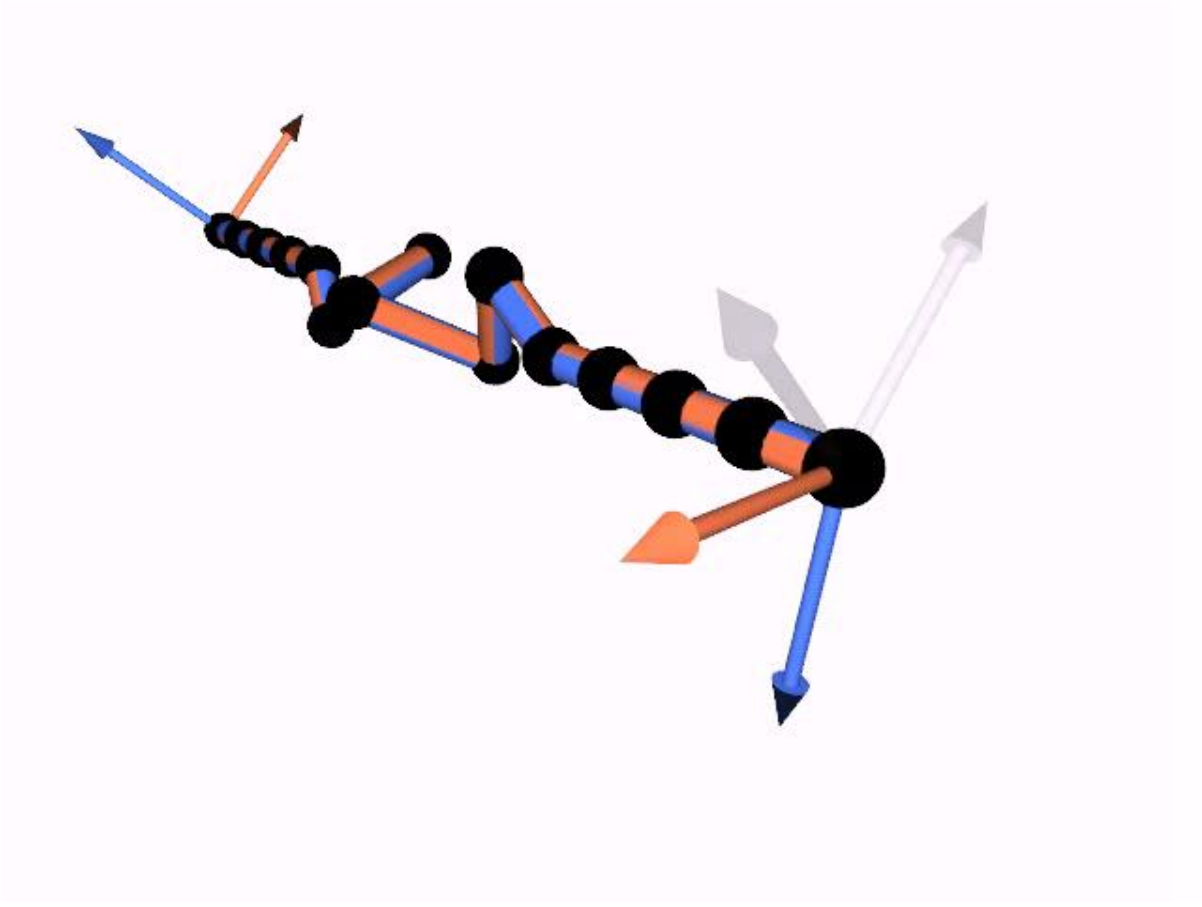
Discrete Curvature



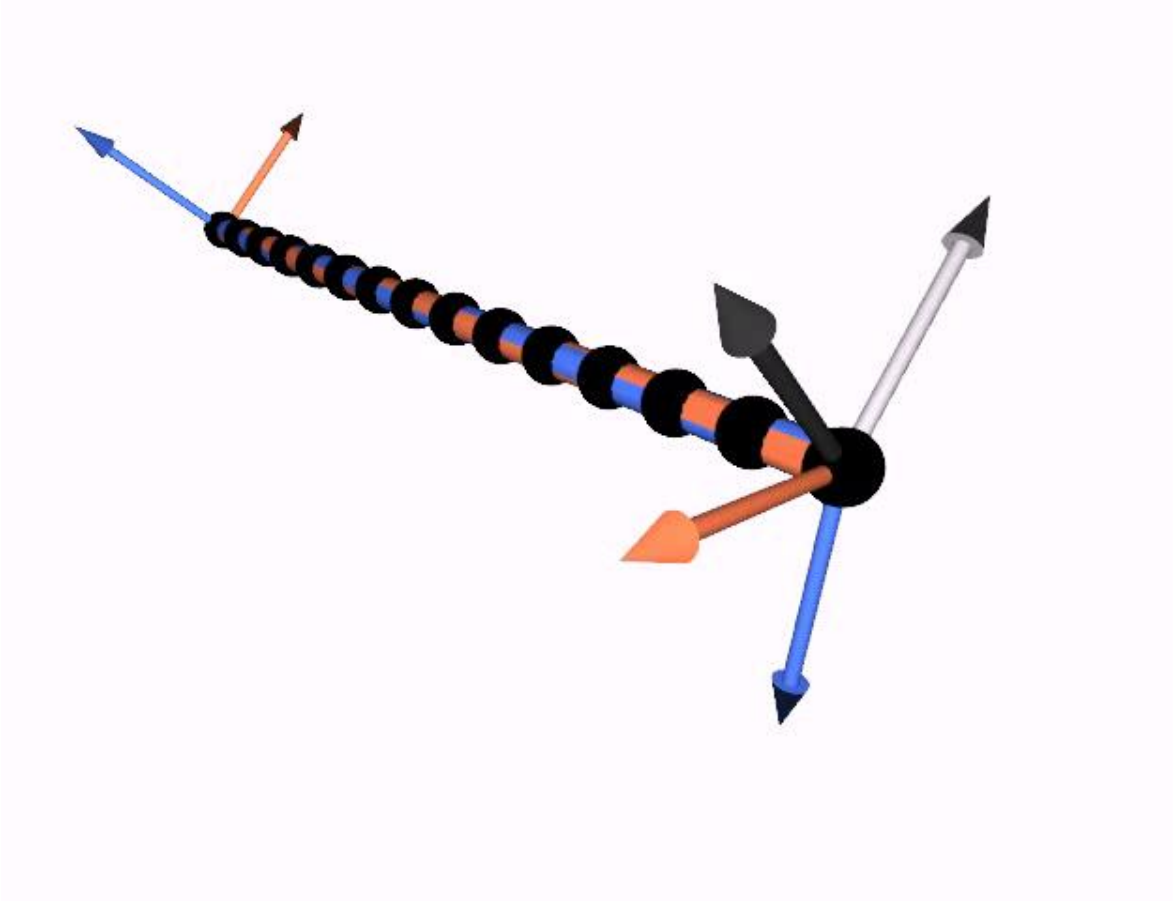
Discrete Curvature



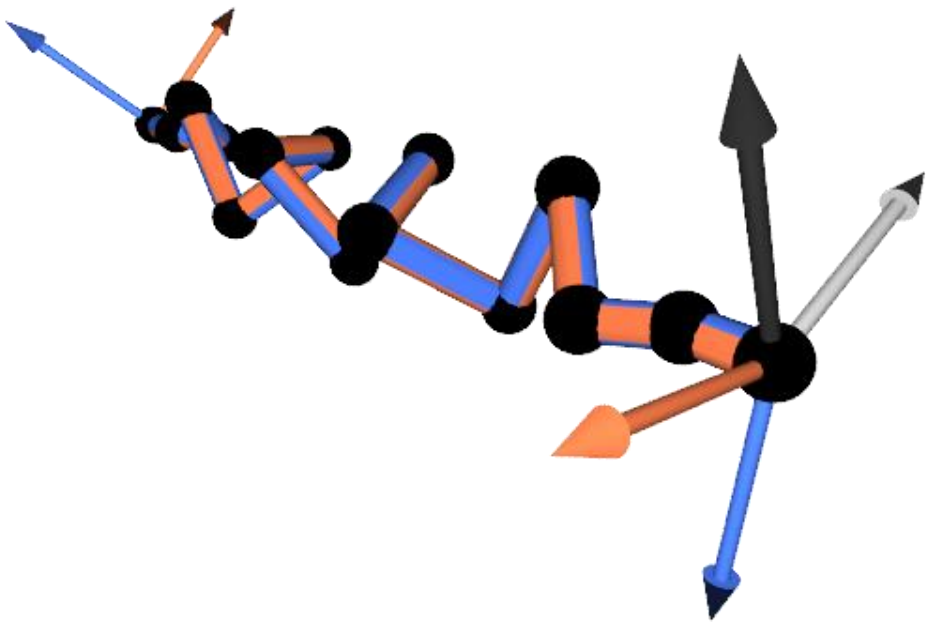
Discrete Curvature



Discrete Curvature

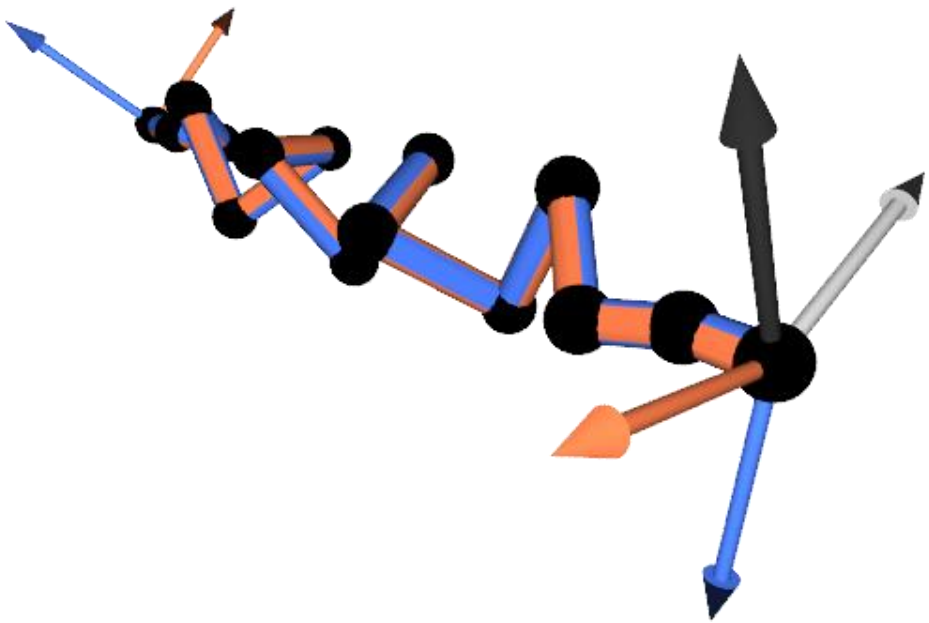


Discrete Curvature



$$-\sum_i 2\tan\left(\frac{\phi_i}{2}\right)\mathbf{b}\cdot\left(\frac{\delta\mathbf{x}_i-\delta\mathbf{x}_{i-1}}{2|\mathbf{x}_i-\mathbf{x}_{i-1}|}+\frac{\delta\mathbf{x}_{i+1}-\delta\mathbf{x}_i}{2|\mathbf{x}_{i+1}-\mathbf{x}_i|}\right)$$

Discrete Curvature



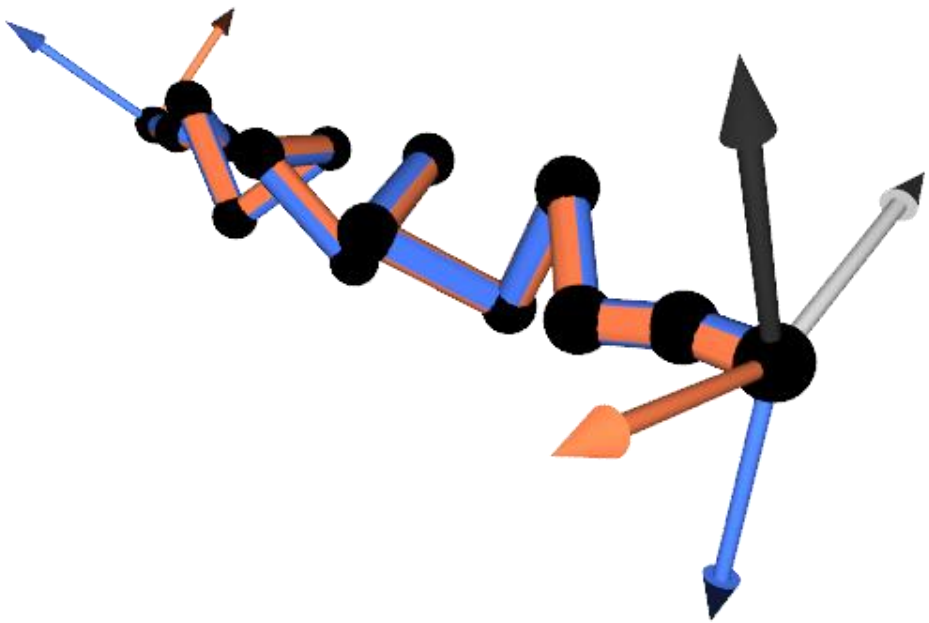
continuous

change in angle

$$-\int (\kappa \mathbf{b}) \cdot (\delta \mathbf{x})' \mathrm{d}s$$

$$-\sum_i 2 \tan\left(\frac{\phi_i}{2}\right) \mathbf{b} \cdot \left(\frac{\delta \mathbf{x}_i - \delta \mathbf{x}_{i-1}}{2|\mathbf{x}_i - \mathbf{x}_{i-1}|} + \frac{\delta \mathbf{x}_{i+1} - \delta \mathbf{x}_i}{2|\mathbf{x}_{i+1} - \mathbf{x}_i|}\right)$$

Discrete Curvature



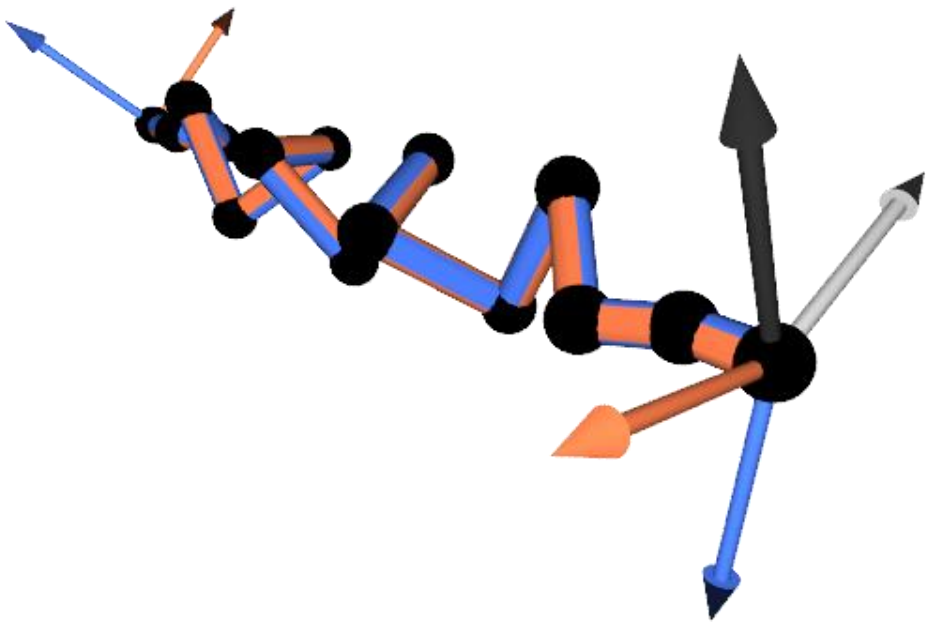
continuous

change in angle

$$-\int (\kappa \mathbf{b}) \cdot (\delta \mathbf{x})' \, \mathrm{d}s$$

$$-\sum_i 2 \tan\left(\frac{\phi_i}{2}\right) \mathbf{b} \cdot \left(\frac{\delta \mathbf{x}_i - \delta \mathbf{x}_{i-1}}{2|\mathbf{x}_i - \mathbf{x}_{i-1}|} + \frac{\delta \mathbf{x}_{i+1} - \delta \mathbf{x}_i}{2|\mathbf{x}_{i+1} - \mathbf{x}_i|} \right)$$

Discrete Curvature



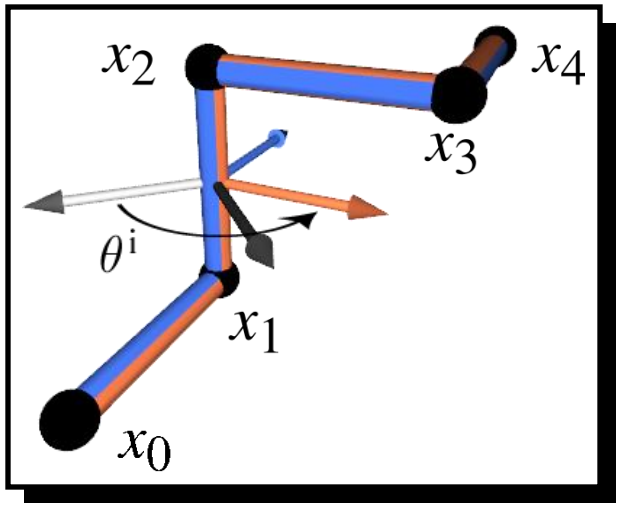
continuous

change in angle

$$-\int \boxed{(\kappa \mathbf{b})} \cdot (\delta \mathbf{x})' \mathrm{d}s$$

$$-\sum_i \boxed{2\tan\left(\frac{\phi_i}{2}\right)\mathbf{b}} \cdot \left(\frac{\delta \mathbf{x}_i - \delta \mathbf{x}_{i-1}}{2|\mathbf{x}_i - \mathbf{x}_{i-1}|} + \frac{\delta \mathbf{x}_{i+1} - \delta \mathbf{x}_i}{2|\mathbf{x}_{i+1} - \mathbf{x}_i|}\right)$$

Discrete Energy

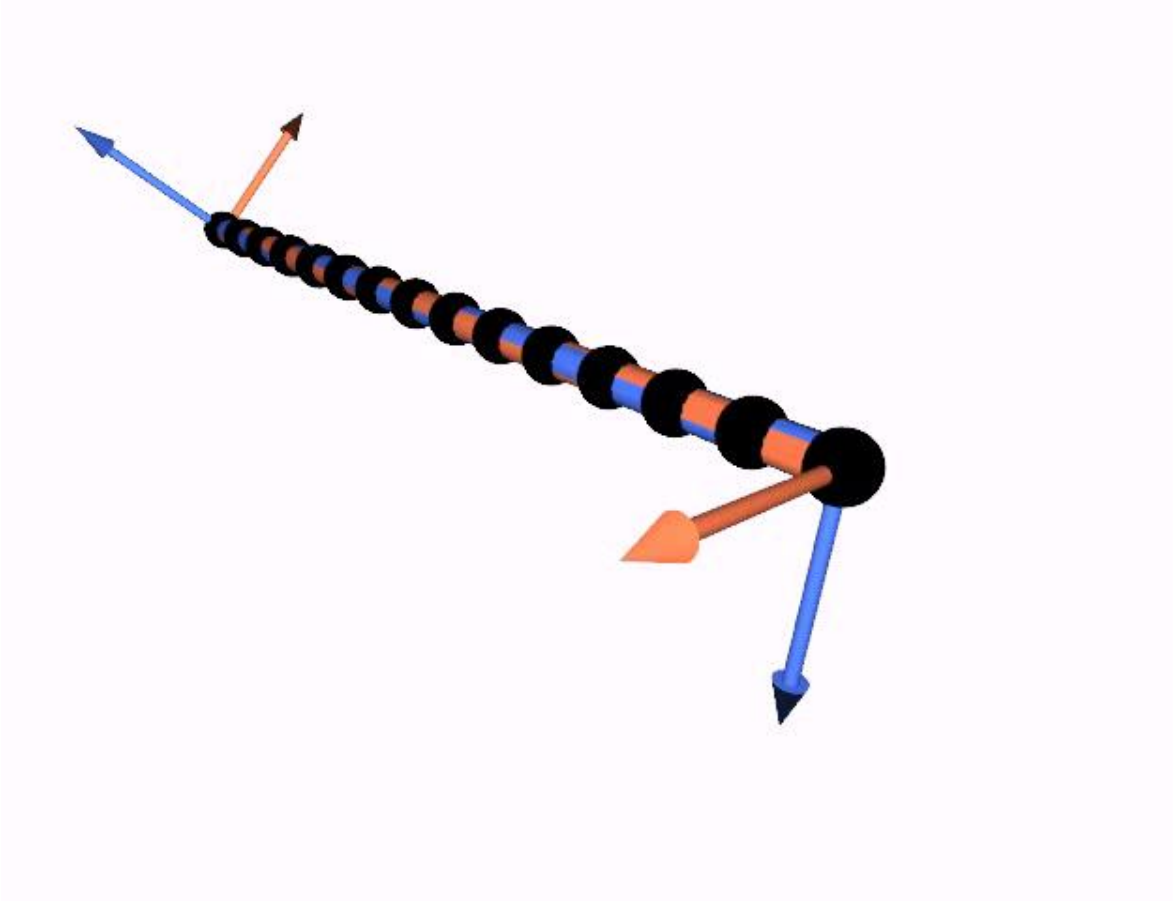


bending twisting

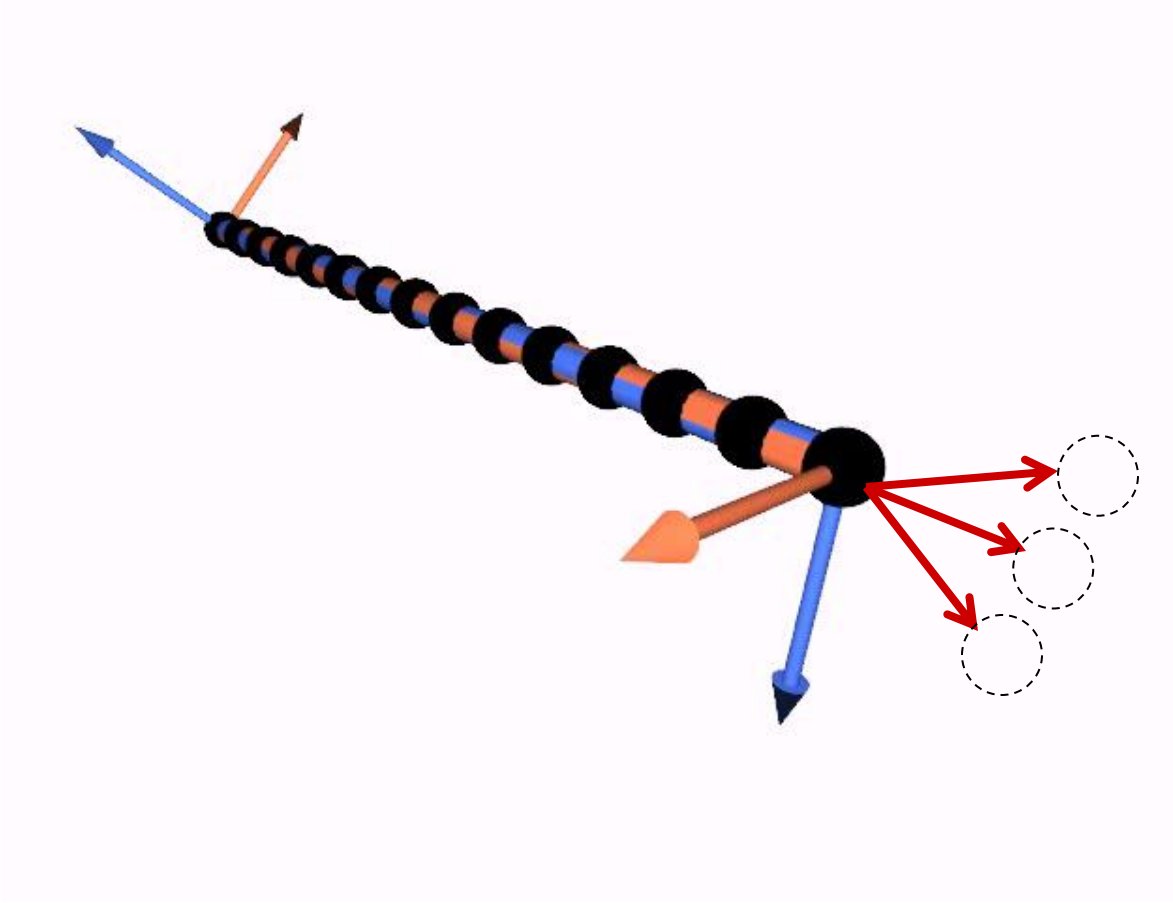
$$\frac{1}{2}\int \alpha \boldsymbol{\kappa}^2 \mathrm{d}s + \frac{1}{2}\int \beta (\theta')^2 \mathrm{d}s$$

$$\kappa_i = 2\tan\left(\frac{\phi_i}{2}\right) \qquad \theta_i' = \frac{\theta^i - \theta^{i-1}}{l_i}$$

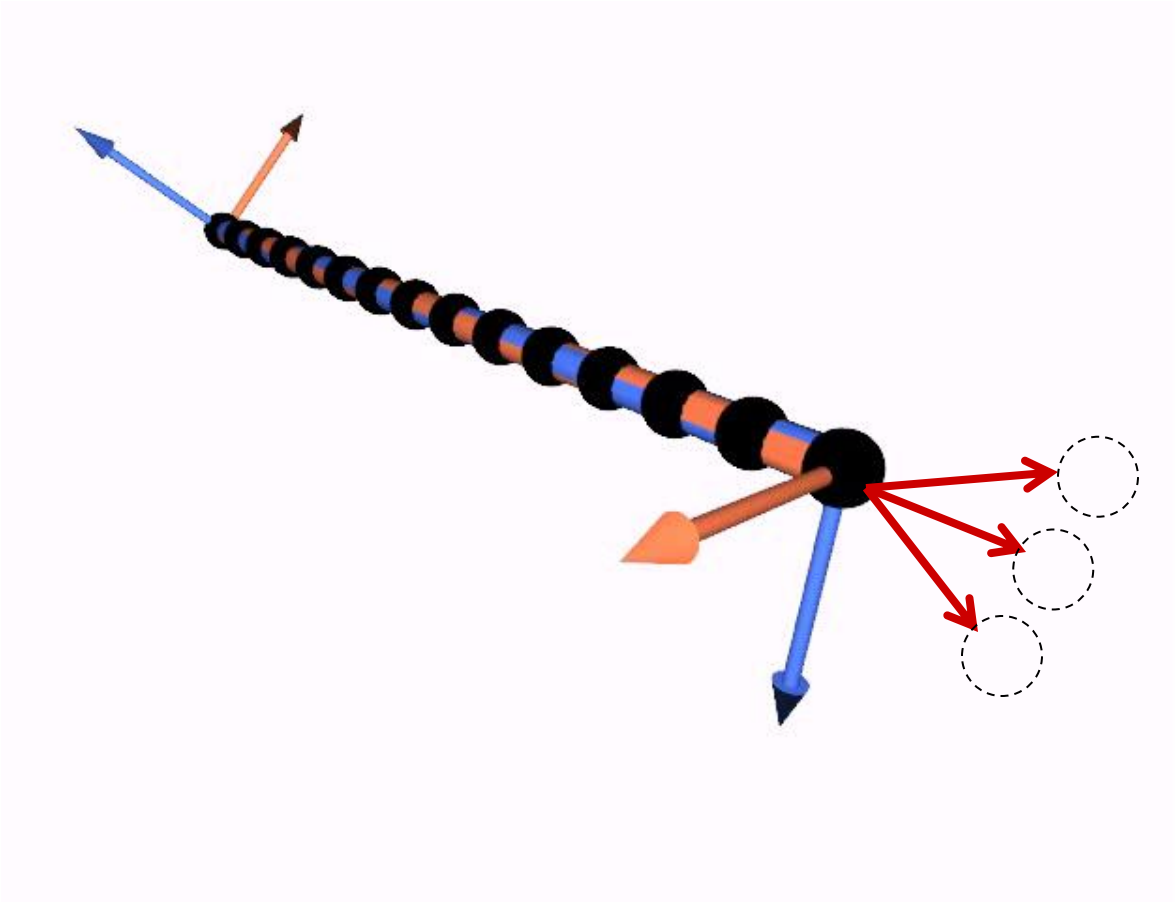
Discrete Curvature

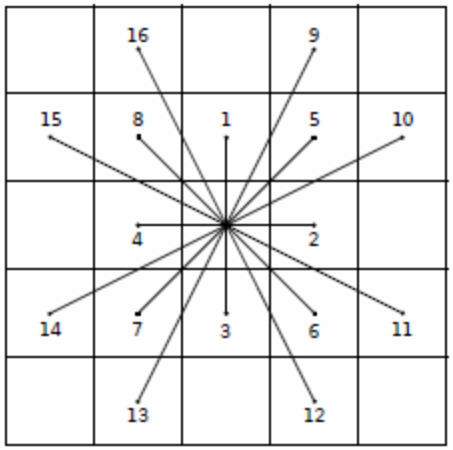


Discrete Curvature

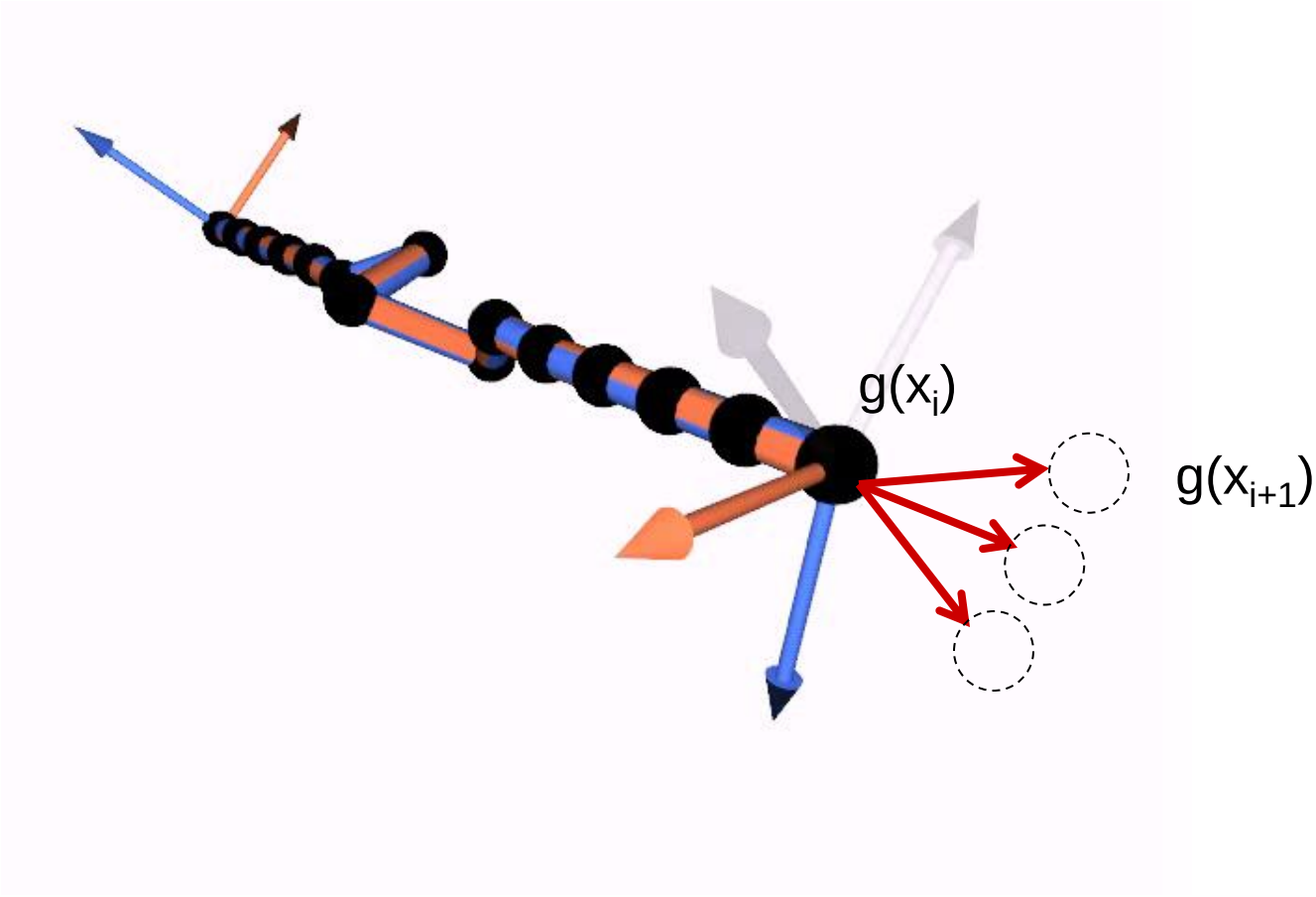


Discrete Neighborhood

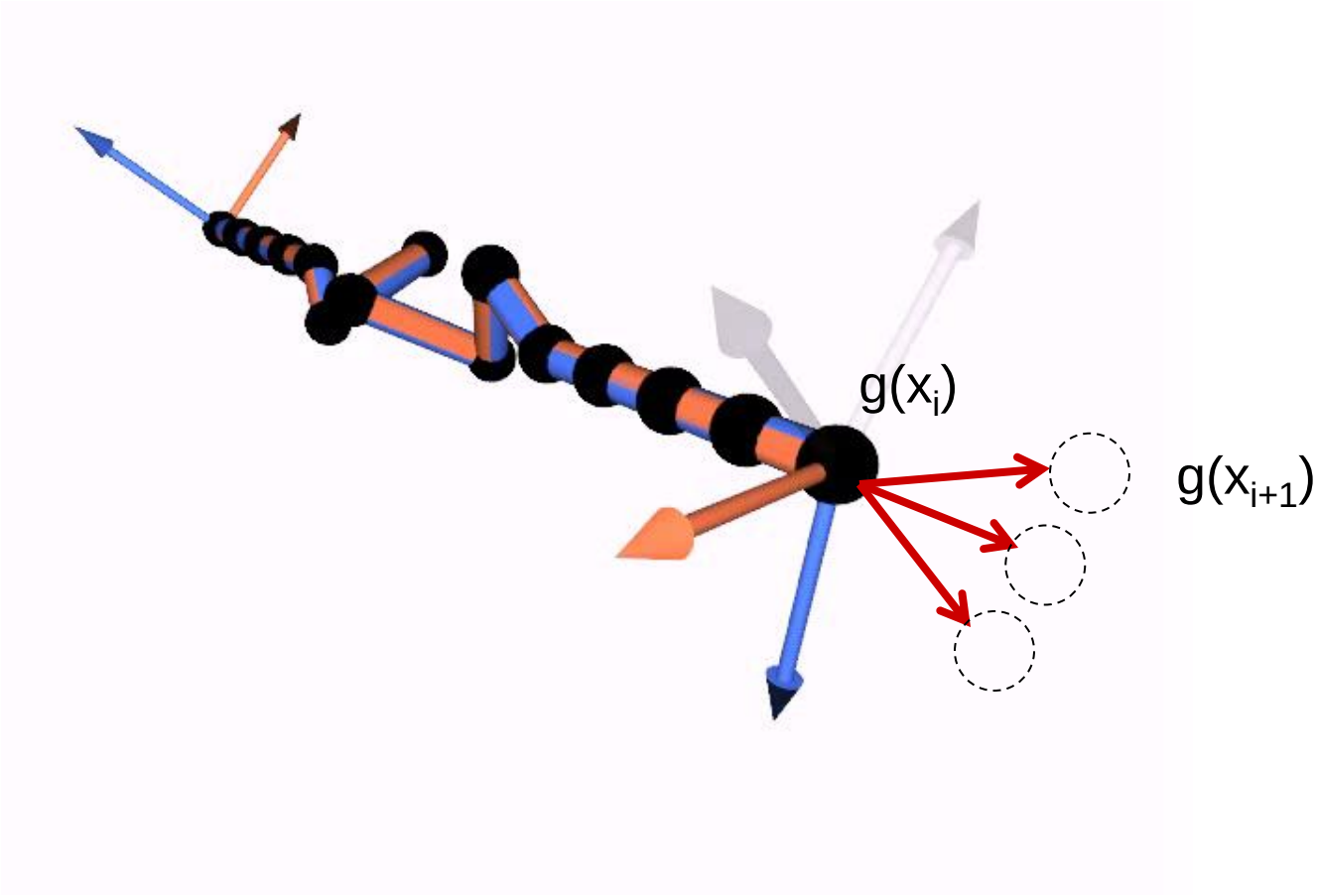




Geodesic Path



Geodesic Path



Thank you