

Bonner Mathematische Schriften

Nr. 395

Ulrich Schlickewei

Hodge classes on self-products of K3 surfaces

Bonn 2009

BONNER MATHEMATISCHE SCHRIFTEN

Nr. 395

Gegenwärtige Herausgeber:

S. Albeverio, H. W. Alt, W. Ballmann, C. - F. Bödigheimer, A. Bovier, S. Conti,
A. Eberle, J. Franke, J. Frehse, M. Griebel, U. Hamenstädt, D. Huybrechts,
H. Koch, P. Koepke, M. Lesch, W. Müller, F. Otto, M. Rapoport, M. Rumpf,
M. Schäl, K. Scherer, J. Schröer, S. Schwede, C. Stroppel, K.-T. Sturm,
O. Venjakob

Begründet von E. Peschl

Fortgeführt von E. Brieskorn, H. Föllmer, G. Harder, G. Hasenjäger,
S. Hildebrandt, F. Hirzebruch, H. Karcher, A. Kunoth, W. Klingenberg,
W. Krull, R. Leis, I. Lieb, F. Pop, E. Sperner, J. Tits, H. Unger, W. Vogel,
H. Werner

Druck: Rheinische Friedrich-Wilhelms-Universität, Bonn
Mathematisches Institut der Universität
Wegelerstr. 10, D-53115 Bonn
email: bibliothek@math.uni-bonn.de
ISSN 0524-045X

Hodge classes on self-products of K3 surfaces

Dissertation

zur Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
Ulrich Schlickewei
aus Freiburg im Breisgau

Bonn 2009



Angefertigt mit der Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Referent: Prof. Dr. D. Huybrechts

2. Referent: Prof. Dr. B. van Geemen

Tag der mündlichen Prüfung: 26. Juni 2009

Diese Dissertation ist auf dem Hochschulschriftenserver der ULB Bonn unter http://hss.ulb.uni-bonn.de/diss_online elektronisch publiziert.

Erscheinungsjahr: 2009.

Summary

This thesis consists of four parts all of which deal with different aspects of Hodge classes on self-products of K3 surfaces.

In the first three parts we present three different strategies to tackle the Hodge conjecture for self-products of K3 surfaces. The first approach is of deformation theoretic nature. We prove that Grothendieck's invariant cycle conjecture would imply the Hodge conjecture for self-products of K3 surfaces. The second part is devoted to the study of the Kuga–Satake variety associated with a K3 surface with real multiplication. Building on work of van Geemen, we calculate the endomorphism algebra of this Abelian variety. This is used to prove the Hodge conjecture for self-products of K3 surfaces which are double covers of \mathbb{P}^2 ramified along six lines. In the third part we show that the Hodge conjecture for $S \times S$ is equivalent to the Hodge conjecture for $\text{Hilb}^2(S)$. Motivated by this, we calculate some algebraic classes on $\text{Hilb}^2(S)$ and on deformations of $\text{Hilb}^2(S)$.

The fourth part includes two additional results related with Hodge classes on self-products of K3 surfaces. The first one concerns K3 surfaces with complex multiplication. We prove that if a K3 surface S has complex multiplication by a CM field E and if the dimension of the transcendental lattice of S over E is one, then S is defined over an algebraic number field. This result was obtained previously by Piatetski-Shapiro and Shafarevich but our method is different. The second additional result says that the André motive $\mathfrak{h}(X)$ of a moduli space of sheaves X on a K3 surface is an object of the smallest Tannakian subcategory of the category of André motives which contains $\mathfrak{h}^2(X)$.

Contents

Summary	1
Introduction	4
1 Deformation theoretic approach	11
1.1 Hodge structures of K3 type	11
1.1.1 Hodge structures	11
1.1.2 Hodge structures of K3 type	14
1.1.3 Endomorphisms of T	16
1.1.4 Mukai's result and K3 surfaces with CM	16
1.1.5 Splitting of T over extension fields	17
1.1.6 Galois action on $T_{\tilde{F}}$	18
1.1.7 Weil restriction	19
1.1.8 The special Mumford–Tate group of T	20
1.2 The variational approach	22
1.2.1 The Hodge locus of an endomorphism	22
1.2.2 Proof and discussion of Theorem 1	24
1.2.3 Twistor lines	28
2 The Kuga–Satake correspondence	31
2.1 Kuga–Satake varieties and real multiplication	31
2.1.1 Clifford algebras	31
2.1.2 Spin group and spin representation	32
2.1.3 Graded tensor product	32
2.1.4 Kuga–Satake varieties	33
2.1.5 Corestriction of algebras	33
2.1.6 The decomposition theorem	36
2.1.7 Galois action on $C(q)_{\tilde{E}}$	37
2.1.8 Proof of the decomposition theorem	39
2.1.9 Central simple algebras	45
2.1.10 An example	46
2.2 Double covers of \mathbb{P}^2 branched along six lines	49
2.2.1 The transcendental lattice	49

2.2.2	Moduli	50
2.2.3	Endomorphisms of the transcendental lattice	51
2.2.4	Abelian varieties of Weil type	52
2.2.5	Abelian varieties with quaternion multiplication	53
2.2.6	The Kuga–Satake variety	53
2.2.7	Proof of Theorem 2	54
3	Hilbert schemes of points on K3 surfaces	56
3.1	The cohomology of the Hilbert square	57
3.1.1	The cohomology ring	57
3.1.2	HC for $S \times S \iff$ HC for $\text{Hilb}^2(S)$	60
3.2	Tautological bundles on the Hilbert square	62
3.2.1	The fundamental short exact sequence	62
3.2.2	The Chern character of $\mathcal{L}^{[2]}$	63
3.2.3	The stability of $\mathcal{L}^{[2]}$	66
3.3	The Fano variety of lines on a cubic fourfold	72
3.3.1	The result of Beauville and Donagi	72
3.3.2	Chern classes of F	73
3.3.3	The image of the correspondence $[Z]_*$	75
3.3.4	The Fano surface of lines on a cubic threefold	76
3.4	Discussion	77
4	Two complementary results	79
4.1	K3 surfaces with CM are defined over number fields	79
4.2	André motives	82
4.2.1	Tensor categories and Tannakian categories	82
4.2.2	André motives	83
4.2.3	Markman’s results	87
4.2.4	The motive of X	90
	Bibliography	93

Introduction

In 1941 in his book [Ho], Hodge formulated a question which since then has become one of the most prominent problems in pure mathematics, known as the Hodge conjecture. His study of the de Rham cohomology of a compact Kähler manifold X had cumulated in the decomposition

$$H^k(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X)$$

which is called the Hodge decomposition. Hodge asked up to which extent the geometry of X is encoded in the cohomology ring $H^*(X, \mathbb{Q})$ together with the decomposition of $H^*(X, \mathbb{C}) = H^*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. He observed that the fundamental class of an analytic subset of codimension k of X is contained in the space

$$B^k(X) := H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X).$$

This led him to

Question 1 (Hodge Conjecture). *Assume that X is projective. Is it true that the space $B^k(X)$ is generated by fundamental classes of codimension k cycles in X ?*

(Hodge actually formulated his question using integral instead of rational coefficients. But work of Atiyah and Hirzebruch and later Kollár showed that this version was too ambitious.)

The answer to the question is known to be affirmative for $k = 0, 1, \dim X - 1, \dim X$. The case $k = 1$ has been proved by Lefschetz using Poincaré's normal functions. This result is known as the Lefschetz theorem on $(1,1)$ classes. By the hard Lefschetz theorem, the theorem on $(1,1)$ classes implies that the Hodge conjecture is true for degree $k = \dim X - 1$. In particular, all smooth, projective varieties of dimension smaller than or equal to 3 satisfy the Hodge conjecture.

Apart from these general facts there are only a few special cases for which the Hodge conjecture has been verified. We list the most prominent of these examples.

- Conte and Murre [CM] showed that the Hodge conjecture is true for uniruled fourfolds. Applying similar ideas, Laterveer [La] was able to extend the result of [CM] to rationally connected fivefolds.

- Mattuck [Mat] showed that on a general Abelian variety all Hodge classes are products of divisor classes. In view of a result of Tate [Ta], the same assertion is true for Abelian varieties which are isogenous to a product of elliptic curves. Later Tankeev [Tk] succeeded to prove that on a simple Abelian variety of prime dimension, all Hodge classes are products of divisor classes. In particular by the Lefschetz theorem on (1,1) classes, all these Abelian varieties satisfy the Hodge conjecture by the Lefschetz theorem on (1,1) classes.

The first examples of Abelian varieties in dimension 4 which carry Hodge classes that are not products of divisor classes were found by Mumford. Later Weil formalized Mumford's approach. He introduced a class of Abelian varieties all of which carry strictly more Hodge classes than products of divisor classes. Nowadays, these varieties are called Abelian varieties of Weil type, we will discuss them below in Section 2.2.4. Moonen and Zarhin [MZ] showed that in dimension less than or equal to five, an Abelian variety either is of Weil type or the only Hodge classes on the variety are products of divisor classes. For Abelian varieties of Weil type the Hodge conjecture remains completely open. Only in special cases it has been verified independently of each other by Schoen and van Geemen (cited as Theorem 2.2.4.1 below).

- Shioda [Shi] has checked the Hodge conjecture for Fermat varieties $Z(X_0^d + \dots + X_n^d) \subset \mathbb{P}^n$ under certain conditions on the degree d and n . The essential tool in his proof is the large symmetry group of these varieties.

- On the product of two surfaces $S_1 \times S_2$, by Poincaré duality, the space of Hodge classes of degree 4 may be identified with the space of \mathbb{Q} -linear homomorphisms

$$H^*(S_1, \mathbb{Q}) \rightarrow H^*(S_2, \mathbb{Q})$$

which respect the degree and the Hodge decomposition.

If S_1 and S_2 are rational surfaces, then $S_1 \times S_2$ is uniruled and thus, in view of [CM] as cited above, the Hodge conjecture is true for $S_1 \times S_2$.

Ramón-Marí [RM] proved that for surfaces S_1, S_2 with $p_g(S_i) = 1, q(S_i) = 2$ (e.g. S_1, S_2 Abelian surfaces) the Hodge conjecture is true for the product $S_1 \times S_2$ (in fact he verifies the Hodge conjecture for a product of n such surfaces).

The next interesting class of surfaces of Kodaira dimension 0 are K3 surfaces. Since K3 surfaces are simply connected, their first and third singular cohomology groups are trivial. Consequently, interesting Hodge classes on a product $S_1 \times S_2$ of two K3 surfaces correspond to homomorphisms of Hodge structures

$$\varphi : H^2(S_1, \mathbb{Q}) \rightarrow H^2(S_2, \mathbb{Q}).$$

A very beautiful and deep result has been proved by Mukai ([Mu1], we quote the precise statement below in Section 1.1.4): Assume that the Picard

number of S_1 is greater than or equal to five. If φ is an isometry with respect to the intersection product, then it is algebraic (i.e. a \mathbb{Q} -linear combination of fundamental classes of codimension 2 subvarieties of $S_1 \times S_2$).

Note that for an isometry φ which induces an isomorphism of the integral cohomology groups, this result is a consequence of the global Torelli theorem for K3 surfaces. In general, Mukai's result is more subtle and it is based upon the theory of moduli spaces of sheaves. In [Mu2], Mukai announced an extension of his result to K3 surfaces with arbitrary Picard number.

But what happens in the case that φ does not preserve the intersection product? Let us restrict ourselves to the special case $S_1 = S_2 = S$. Write $T(S) \subset H^2(S, \mathbb{Q})$ for the orthogonal complement of the rational Néron–Severi group $\text{NS}(S)$. Then an endomorphism $\varphi : H^2(S, \mathbb{Q}) \rightarrow H^2(S, \mathbb{Q})$ which preserves the Hodge decomposition, splits as a sum $\varphi = \varphi_t + \varphi_n$ where $\varphi_t : T(S) \rightarrow T(S)$ and $\varphi_n : \text{NS}(S) \rightarrow \text{NS}(S)$. By the Lefschetz theorem on (1,1) classes, we may infer that φ_n is algebraic. Therefore, the Hodge conjecture for $S \times S$ reduces to

Question 2 (Hodge conjecture for self-products of K3 surfaces). *Is it true that the space $\text{End}_{\text{Hdg}}(T(S))$ of endomorphisms of $T(S)$ which respect the Hodge decomposition is generated by algebraic classes?*

In this thesis we present three different strategies to tackle this question.

The departing point are the famous results of Zarhin which give a complete description of the algebra $E(S) := \text{End}_{\text{Hdg}}(T(S))$. In [Z] it is shown that $E(S)$ is an algebraic number field which can be either totally real (in this case we say that S has *real multiplication*) or a CM field (we say that S has *complex multiplication*). It was pointed out by Morrison [Mo] that Mukai's results imply the Hodge conjecture for self-products of K3 surfaces with complex multiplication. Consequently, we will concentrate on K3 surfaces with real multiplication.

The first approach in this thesis is of deformation theoretic nature. First we consider projective deformations. Our main result here is

Theorem 1. *Let S be a K3 surface with real multiplication by a totally real number field $E = \text{End}_{\text{Hdg}}(T(S))$. Let $\varphi \in E$. Then there exist a smooth, projective morphism of smooth, quasi-projective, connected varieties $\pi : \mathcal{X} \rightarrow B$, a base point $0 \in B$ with fiber $\mathcal{X}_0 \simeq \pi^{-1}(0) = S$ and a dense subset $\Sigma \subset B$ with the following properties:*

- (i) φ is monodromy-equivariant,
- (ii) for each $s \in \Sigma$ the homomorphism $\varphi_s \in \text{End}_{\mathbb{Q}}(H^2(\mathcal{X}_s, \mathbb{Q}))$, obtained by parallel transport of φ , is algebraic.

This result reduces the Hodge conjecture for $S \times S$ to Grothendieck's invariant cycle conjecture. (This conjecture is recalled in Section 1.2.2.) Such

a reduction has been derived previously by Y. André [An1] (see also [De]). His arguments rely heavily on the Kuga–Satake correspondence, whereas we give a more direct approach.

It is known, again by results of André [An4], that for a given family of products of surfaces, Grothendieck’s invariant cycle conjecture follows from the standard conjecture B for a smooth compactification of the total space of the family. (We recall in Section 4.2.2 the statement of the standard conjecture B). Therefore our result implies that, in order to prove the Hodge conjecture for self-products of K3 surfaces, it would suffice to prove the Lefschetz standard conjecture for total spaces of pencils of self-products of K3 surfaces. However, this seems to be a hard problem.

There is another distinguished class of deformations of a K3 surface S , the twistor lines. Each Kähler class on S can be represented by the Kähler form of a Hyperkähler metric which gives rise to a two-sphere of complex structures on the differentiable fourfold underlying S . In this way one obtains a deformation of S parametrized by \mathbb{P}^1 . Verbitsky [Ve1] found a very nice criterion which decides when a subvariety N of S is compatible with a Hyperkähler structure on S (such a subvariety is called *trianalytic*). Verbitsky [Ve2] could also derive a criterion for a complex vector bundle \mathcal{E} on S to be compatible with a Hyperkähler structure (in this case, \mathcal{E} is called *hyperholomorphic*). The precise statements are recalled below in Theorem 1.2.3.1. We study the question whether real or complex multiplication can deform along twistor lines. The answer is negative for complex multiplication. In contrast to this, we prove that if S has real multiplication by a real quadratic number field E and if the Picard number of S is greater than or equal to three, then there exist twistor lines along which the generator φ of E (extended appropriately by an endomorphism of the Néron–Severi group) remains an endomorphism of Hodge structures. Each Hyperkähler structure on S induces such a structure on $S \times S$. It would be very interesting to represent the class φ by a trianalytic subvariety of $S \times S$ or by a hyperholomorphic vector bundle on $S \times S$.

In the second part of this thesis we concentrate on the Kuga–Satake correspondence, a very useful tool in the theory of K3 surfaces which associates to a K3 surface S an (isogeny class of an) Abelian variety A such that $H^2(S, \mathbb{Q})$ is contained in $H^2(A \times A, \mathbb{Q})$. This correspondence shows up in many important results on K3 surfaces, cf. for example Deligne’s proof of the Weil conjecture for K3 surfaces. Unfortunately, the construction of the Kuga–Satake variety is purely Hodge-theoretic and we don’t know in general how to relate A and S geometrically.

We reformulate and improve slightly a result of van Geemen [vG4] which gives us a decomposition of the Kuga–Satake variety A of a K3 surface S with real multiplication by a totally real number field E . This allows us

to identify the endomorphism algebra of A with the corestriction to \mathbb{Q} of a Clifford algebra over E . We give a concrete example where we calculate this corestriction explicitly.

Next, we study one of the few families of K3 surfaces for which a geometric explanation of the Kuga–Satake correspondence is available in the literature by a result of Paranjape [P]. This is the four-dimensional family of double covers of \mathbb{P}^2 which are ramified along six lines. Building on the decomposition of the Kuga–Satake variety we derive

Theorem 2. *Let S be a K3 surface which is a double cover of \mathbb{P}^2 ramified along six lines. Then the Hodge conjecture is true for $S \times S$.*

As pointed out by van Geemen [vG4], there are one-dimensional subfamilies of the family of such double covers with real multiplication by a quadratic totally real number field. In conjunction with our Theorem 2, this allows us to produce examples of K3 surfaces S with non-trivial real multiplication for which $\text{End}_{\text{Hdg}}(T(S))$ is generated by algebraic classes. We could not find examples of this type in the existing literature.

The third part of this thesis is of a more concrete nature. Using Mukai’s result we show

Proposition 3. *Let S be a K3 surface. Then the Hodge conjecture is true for $S \times S$ if and only if it is true for $\text{Hilb}^2(S)$.*

The interest in Proposition 3 stems from a result of Beauville and Donagi which reads as follows: Let S be a general K3 surface of degree 14 in \mathbb{P}^8 . Then there exists a smooth cubic fourfold $Y \subset \mathbb{P}^5$ such that the Fano variety $F(Y)$ parameterizing lines contained in Y is isomorphic to $\text{Hilb}^2(S)$.

This twofold description of $\text{Hilb}^2(S)$ as a moduli space allows us to use the geometry of S and of Y to produce algebraic cycles on $\text{Hilb}^2(S) \simeq F(Y)$. Along this line we calculate the Chern character of the tautological bundle $\mathcal{L}^{[2]}$ on $\text{Hilb}^2(S)$ associated with a line bundle $\mathcal{L} \in \text{Pic}(S)$. If $h^0(\mathcal{L}) \geq 2$, then $\mathcal{L}^{[2]}$ is shown to be stable on $\text{Hilb}^2(S)$ with respect an appropriate polarization. It is interesting to have examples of stable vector bundles in view of Verbitsky’s criterion which allows to control deformations of vector bundles along twistor lines. Finally, we calculate the fundamental classes of some natural surfaces in $F(Y)$ which are induced by Y .

In addition to the above mentioned results we include in this thesis two further theorems which came out on the way. Even if they are not directly related to Question 2 they might have some interest and some beauty on their own.

The first one deals with K3 surfaces with complex multiplication.

Theorem 4. *Let S be a K3 surface with complex multiplication by a CM field E . Assume that $m = \dim_E T(S) = 1$. Then S is defined over an algebraic number field.*

This was known by a classical result of Piatetski-Shapiro and Shafarevich [PSS], more recently it was proved by Rizov as a part of the main theorem on complex multiplication for K3 surface [Ri]. Our approach is different, it relies upon the study of the Hodge locus of an endomorphism of a K3 type Hodge structure and upon Mukai's result.

The second additional theorem is concerned with André's category of motives. We use Markman's result on the monodromy group of moduli spaces of sheaves on K3 surfaces to get

Theorem 5. *Let Y be a projective deformation of a smooth moduli space of sheaves on a K3 surface S . Then the André motive $\mathfrak{h}(Y)$ is an object of $\langle \mathfrak{h}^2(Y) \rangle$, the smallest Tannakian subcategory of the category of motives containing $\mathfrak{h}^2(Y)$.*

This can be seen as a manifestation of the general principle that the geometry of a Hyperkähler variety is governed by its second cohomology and it has some interesting consequences. Among these we mention that all Hodge classes on Y are absolute in the sense of Deligne. Moreover, the Hodge conjecture for Y would follow from the standard conjecture B for all smooth, projective varieties.

Organization of the thesis

In **Chapter 1** we present the deformation theoretic approach.

Section 1.1 settles basic notions which will play a role throughout the whole thesis. We recall the notion of a Hodge structure of (primitive) K3 type and we review Zarhin's famous result on the endomorphism algebra of such a Hodge structure. Next, we recall how Mukai's result implies that Question 2 has a positive answer for K3 surfaces with complex multiplication. The remainder of the Section deals with the linear algebra of a Hodge structure of primitive K3 type with real multiplication.

In Section 1.2 we determine the Hodge locus in the period domain of an endomorphism of a Hodge structure of K3 type. Building on the computations of the first section we then prove Theorem 1. Subsection 1.2.3 is devoted to the study of twistor deformations.

Chapter 2 deals with the Kuga–Satake correspondence and its applications to double covers of \mathbb{P}^2 ramified along six lines.

In Section 2.1 we review the definition of the Kuga–Satake variety, we recall the definition of the corestriction of an algebra and we prove the decomposition theorem for Kuga–Satake varieties of Hodge structures of K3 type with real multiplication. Finally, we give an explicit example.

Section 2.2 studies double covers of \mathbb{P}^2 which are ramified along six lines. We recall a result of Lombardo which says that the Kuga–Satake variety of such a K3 surface is of Weil type with discriminant 1. Combining this with

Paranjape’s result, which establishes the algebraicity of the Kuga–Satake variety, and with Schoen’s and van Geemen’s result on Abelian fourfolds of Weil type, we derive Theorem 2.

In **Chapter 3** we discuss Hilbert schemes of K3 surfaces. In Section 3.1, we review the cohomology ring of $\text{Hilb}^2(S)$ to prove Proposition 3. The remainder of the Chapter is devoted to concrete calculations, first on the Hilbert scheme in Section 3.2, then on the Fano variety of lines in Section 3.3. Finally we discuss the results in Section 3.4.

Chapter 4 consists of two sections.

Section 4.1 is devoted to the proof of Theorem 4.

In Section 4.2 we review in some detail the category of André motives and we state his deformation principle. Next, we quote some of the results of Markman’s work on the monodromy group of moduli spaces of sheaves on K3 surfaces. Finally, we reformulate them in André’s language and prove Theorem 5.

Danksagung. Zuvorderst möchte ich meinem Doktorvater Professor Daniel Huybrechts danken. Er stand mir stets mit guten Ideen, mit wertvollem Rat und mit viel Geduld zur Seite. Sein starkes Interesse an meiner Arbeit wie auch seine kontinuierlichen Ermutigungen waren von großem Wert für mich.

Weiterhin bin ich Professor Bert van Geemen zu großer Dankbarkeit verpflichtet. Während meines einmonatigen Aufenthalts in Mailand im Frühjahr 2008 gab er mir eine Fülle von fruchtbaren Anregungen.

Für hilfreiche Gespräche und Diskussionen möchte ich zudem Meng Chen, Moritz Groth, Emanuele Macrì, Sven Meinhardt, Ernesto Mistretta, Arvid Perego, Luca Scala und Paolo Stellari danken.

Während der Anfertigung dieser Arbeit wurde ich von der Studienstiftung des Deutschen Volkes, von der Bonn International Graduate School in Mathematics und vom SFB/TR 45 “Periods, moduli spaces and arithmetic of algebraic varieties” der Deutschen Forschungsgemeinschaft finanziell unterstützt.

Chapter 1

Deformation theoretic approach

1.1 Hodge structures of K3 type

In this section we introduce the notion of a Hodge structure of K3 type which mimics the properties of the second cohomology of a K3 surface but also of an irreducible symplectic variety. We review Zarhin's results on the endomorphism algebra of a Hodge structure of K3 type which distinguish the cases of real and complex multiplication. Next, we discuss Mukai's results the Shafarevich conjecture for K3 surfaces. They solve as a special case the Hodge conjecture for self-products of K3 surfaces with complex multiplication and with Picard number greater than or equal to five. Mukai announces that the same ideas work for any Picard number by using the theory of moduli spaces of twisted sheaves. For this reason, we focus in the sequel on the case of Hodge structures of K3 type with real multiplication, we describe a splitting over some algebraic number field and we review Zarhin's description of the special Mumford–Tate group.

1.1.1 Hodge structures

Denote by $U(1)$ the one-dimensional unitary group which is a real algebraic group. We recall some basic notions on Hodge structures.

- A (*rational*) *Hodge structure of weight k* is a finite-dimensional \mathbb{Q} -vector space T together with a morphism of real algebraic groups

$$h : U(1) \rightarrow \mathrm{GL}(T)_{\mathbb{R}}$$

which after tensoring with \mathbb{C} becomes equivalent to the diagonal representation $z \mapsto \mathrm{diag}(\dots, z^p \bar{z}^{k-p}, \dots)$. For $p, q \in \mathbb{Z}$ with $p + q = k$ denote by

$$T^{p,q} := \{t \in T_{\mathbb{C}} \mid h(z)t = z^p \bar{z}^q t \quad \forall z \in U(1)(\mathbb{R})\}.$$

Then we get a natural decomposition

$$T_{\mathbb{C}} = \bigoplus_{p,q} T^{p,q} \quad (1.1)$$

and the spaces $T^{p,q}$ satisfy the condition

$$T^{p,q} = \overline{T^{q,p}}. \quad (1.2)$$

Equivalently, a Hodge structure of weight k can be defined as a finite-dimensional \mathbb{Q} -vector space T together with a decomposition

$$T_{\mathbb{C}} = \bigoplus_{p+q=k} T^{p,q}$$

of $T_{\mathbb{C}}$ such that the $T^{p,q}$ satisfy (1.2).

Another equivalent definition is to ask for a finite-dimensional \mathbb{Q} -vector space T together with a decreasing filtration $(F^p T_{\mathbb{C}})_{p \in \mathbb{Z}}$ of $T_{\mathbb{C}}$ such that

$$F^p T \oplus \overline{F^{k+1-p} T} = T_{\mathbb{C}} \text{ for all } p \in \mathbb{Z} \quad (1.3)$$

- A Hodge structure of weight k is *effective* if in the decomposition (1.1) $T^{p,q} = 0$ unless $p \geq 0$ and $q \geq 0$.

- There are an obvious notions of tensor products, morphisms and duals of Hodge structure: take the corresponding notions for \mathbb{Q} -vector spaces and ask for compatibility with the $U(1)$ -representation after tensoring with \mathbb{R} . Note in particular that with this definition, any morphism between two Hodge structures of different weight is zero.

If (T, h) and (T', h') are two Hodge structures, the space of homomorphisms of Hodge structures will be denoted by $\text{Hom}_{\text{Hdg}}(T, T')$.

A Hodge structure is *irreducible* if it has no non-trivial sub-Hodge structures.

- In order to compare Hodge structures of different weights, we introduce the *Tate Hodge structure* $\mathbb{Q}(1)$ by defining $\mathbb{Q}(1) = \mathbb{Q}$ with the representation $h_{\mathbb{Q}(1)} : U(1) \rightarrow \text{GL}(\mathbb{R}) = \mathbb{R}^*$, $z \mapsto (z\bar{z})^{-1}$. For all $m \in \mathbb{Z}$ we set $\mathbb{Q}(m) := \mathbb{Q}(1)^{\otimes m}$ and for a Hodge structure (T, h) we set $T(m) := T \otimes_{\mathbb{Q}} \mathbb{Q}(m)$. Note that if (T, h) is of weight k then $T(m)$ has weight $k - 2m$. Thus, the Tate Hodge structure allows us to discuss morphisms between two Hodge structures whose weights have the same parity by twisting one of the two as above.

- Assume that (T, h) has weight $k = 2p$. A *Hodge class* is a class $v \in T \cap T^{p,p}$. The set of Hodge classes of (T, h) is denoted by $B(T)$.

- The *special Mumford–Tate group* or *Hodge group* $\text{SMT}(T)$ of (T, h) is the smallest algebraic subgroup of $\text{GL}(T)$ (defined over \mathbb{Q}) such that

$h(\mathrm{U}(1)) \subset \mathrm{SMT}(T)_{\mathbb{R}}$. This group has the property that sub-representations of $T^{\otimes k} \otimes (T^{\vee})^{\otimes l}$ for $k \geq 0, l \geq 0$ are precisely the sub-Hodge structures of $T^{\otimes k} \otimes (T^{\vee})^{\otimes k}$ (see [Go, Prop. B54]). Here the representation of $\mathrm{SMT}(T)$ on the tensor product is the natural one induced by the inclusion $\mathrm{SMT}(T) \subset \mathrm{GL}(T)$. In particular we obtain

$$B(T^{\otimes k} \otimes (T^{\vee})^{\otimes l}) = (T^{\otimes k} \otimes (T^{\vee})^{\otimes l})^{\mathrm{SMT}(T)}. \quad (1.4)$$

Moreover, a Hodge structure (T, h) is irreducible if and only if the representation of $\mathrm{SMT}(T)$ is so.

- A *polarization* of a weight k Hodge structure (T, h) is a morphism of Hodge structures $q : T \otimes T \rightarrow \mathbb{Q}(-k)$ such that $(-1)^{k(k-1)/2} q(*, h(i)*)$ is a symmetric, positive definite bilinear form on $T_{\mathbb{R}}$.

- If (T, h) is a polarizable Hodge structure (that is there exists a polarization) then the special Mumford–Tate group $\mathrm{SMT}(h)$ is reductive (see e.g. [vG3], Thm. 3.5).

Example. Let X be a compact Kähler manifold. Then $H^k(X, \mathbb{Q})$ carries a Hodge structure of weight k .

If X is projective, then for $k \leq \dim X$ by the Hodge–Riemann bilinear relations, the primitive cohomology $H_p^k(X, \mathbb{Q})$ with respect to any rational Kähler class ω carries a Hodge structure which can be polarized by the bilinear form

$$\begin{aligned} q : H_p^k(X, \mathbb{Q}) \times H_p^k(X, \mathbb{Q}) &\rightarrow \mathbb{Q} \\ (\alpha, \beta) &\mapsto \int_X \alpha \wedge \beta \wedge \omega^{\dim X - k}. \end{aligned}$$

If X is a compact Kähler manifold of dimension n , then Poincaré duality yields an isomorphism $H^k(X, \mathbb{Q})^* \simeq H^{n-k}(X, \mathbb{Q})(n)$. Let Y be another compact Kähler manifold and let $l = k + 2r$ for some $r \in \mathbb{Z}$. Then we get an identification

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Q}}(H^k(X, \mathbb{Q})(-r), H^l(Y, \mathbb{Q})) &\simeq H^{n-k}(X, \mathbb{Q})(n+r) \otimes H^l(Y, \mathbb{Q}) \\ &\subset H^{n-k+l}(X \times Y, \mathbb{Q})(n+r). \end{aligned}$$

Here, homomorphisms of Hodge structures are identified with

$$B(H^{n-k}(X, \mathbb{Q}) \otimes H^l(Y, \mathbb{Q})(n+r)) \subset B(H^{n-k+l}(X \times Y, \mathbb{Q})(n+r)).$$

If $Z \subset X$ is an analytic subset of codimension k , then the fundamental class of Z is a class

$$[Z] \in B(H^{2k}(X, \mathbb{Q})).$$

Any rational linear combination in $B(H^{2k}(X, \mathbb{Q}))$ of fundamental classes of analytic subsets is called an *algebraic class*.

Hodge conjecture. *Let X be a smooth, projective variety over \mathbb{C} . Then for all k the space $B(H^{2k}(X, \mathbb{Q}))$ consists of algebraic classes.*

By the Lefschetz theorem on $(1, 1)$ classes, it is known that all rational classes of type $(1, 1)$ and of type $(\dim X - 1, \dim X - 1)$ are algebraic. Apart from this result the conjecture has been checked only in a few special cases (see the Introduction).

1.1.2 Hodge structures of K3 type

Let S be a complex, projective K3 surface, that is a simply connected, compact complex surface with $K_S = 0$. By the fact that $H^1(S, \mathbb{Q}) = H^3(S, \mathbb{Q}) = 0$ and by the Künneth formula we have

$$B(H^4(S \times S, \mathbb{Q})) = B(H^2(S, \mathbb{Q}) \otimes H^2(S, \mathbb{Q})) \\ \oplus (H^0(S, \mathbb{Q}) \otimes H^4(S, \mathbb{Q})) \oplus (H^4(S, \mathbb{Q}) \otimes H^0(S, \mathbb{Q})).$$

The last two summands are spanned by the Künneth factors of the diagonal which are algebraic for surfaces. Therefore, the Hodge conjecture for $S \times S$ is reduced to the question whether classes in $B(H^2(S, \mathbb{Q}) \otimes H^2(S, \mathbb{Q}))$ are algebraic. This space is identified (up to a Tate twist) via the example in 1.1.1 with $\text{End}_{\text{Hdg}}(H^2(S, \mathbb{Q}))$. We are now going to formalize this situation in order to study the endomorphism algebra in more detail.

Let S be a (not necessarily projective) K3 surface. Then, the second cohomology $H^2(S, \mathbb{Q})$ carries a quadratic form

$$q : H^2(S, \mathbb{Q}) \times H^2(S, \mathbb{Q}) \rightarrow \mathbb{Q} \quad (\alpha, \beta) \mapsto \int_S \alpha \wedge \beta.$$

Fix a Kähler class $\omega \in \mathcal{K}_S$ (any K3 surface is of Kähler type by a result of Siu). By the Hodge–Riemann bilinear relations, q has the property that $-q(*, h(i)*)$ is positive definite on $H_p^2(S, \mathbb{R}) = \omega^\perp$. This motivates the

Definition 1.1.2.1. A *Hodge structure of K3 type* consists of a quadratic \mathbb{Q} -vector space (W, q) of dimension r such that the signature of q is $(3+, (r - 3)-)$ together with an effective Hodge structure $h : \text{U}(1) \rightarrow \text{GL}(W_{\mathbb{R}})$ of weight 2 with the following properties:

- (i) $\dim_{\mathbb{C}} W^{2,0} = 1$,
- (ii) $q : W \otimes W \rightarrow \mathbb{Q}(-2)$ is a morphism of Hodge structures,
- (iii) for all $\omega \in W^{1,1} \cap W_{\mathbb{R}}$ with $q(\omega) > 0$ (such an ω exists by the hypothesis on the signature), the quadratic form $-q(*, h(i)*)$ is positive definite on ω^\perp .

If ω in (ii) can be chosen in W then the Hodge structure of K3 type is said to be *algebraic*.

Let (W, h, q) be a Hodge structure of K3 type. Then the subspaces $W^{2,0} \oplus W^{0,2} \subset W_{\mathbb{C}}$ and $W^{1,1}$ are defined over \mathbb{R} . The symmetric bilinear form q is positive definite on $(W^{2,0} \oplus W^{0,2})_{\mathbb{R}}$ and negative definite on $\omega^{\perp} \cap W_{\mathbb{R}}^{1,1}$ (because $C = h(i)$ acts as $-\text{id}$ on $(T^{2,0} \oplus W^{0,2})_{\mathbb{R}}$ and as id on $(W^{1,1})_{\mathbb{R}}$). Note that a Hodge structure of K3 type cannot be polarized by q because this form is indefinite on $W_{\mathbb{R}}^{1,1}$. However, if (W, h, q) is an algebraic Hodge structure of K3 type and if $\omega \in W^{1,1} \cap W$ with $q(\omega) > 0$, then $\omega^{\perp} \subset W$ is polarized by q . This leads us to

Definition 1.1.2.2. A polarized Hodge structure of primitive K3 type consists of a quadratic \mathbb{Q} -vector space (T, q) and a Hodge structure $h : \text{U}(1) \rightarrow \text{GL}(T)_{\mathbb{R}}$ with the properties:

- (i) $\dim_{\mathbb{C}} T^{2,0} = 1$,
- (ii) q is polarization of (T, h) .

For a Hodge structure of K3 type (W, h, q) , denote by $\text{NS}(W)$ the set of $(1, 1)$ -classes, that is $\text{NS}(W) := B(W, h)$. Let $T(W) := \text{NS}(W)^{\perp}$ be the (rational) transcendental lattice of (W, h) . Assume that W is an algebraic Hodge structure of K3 type. Then we have an orthogonal decomposition

$$W \simeq \text{NS}(W) \oplus^{\perp} T(W). \quad (1.5)$$

This is a decomposition of Hodge structures, meaning that $h = (1, h')$ where $1 : \text{U}(1) \rightarrow \text{GL}(\text{NS}(W))_{\mathbb{R}}$ is the trivial homomorphism mapping all $z \in \text{U}(1)$ to the identity and $h' : \text{U}(1) \rightarrow \text{GL}(T(W))_{\mathbb{R}}$. Note that $T(W)$ is a polarized Hodge structure of primitive K3 type.

Example. By definition, the second cohomology $H^2(S, \mathbb{Q})$ of a K3 surface S carries a Hodge structure of K3 type. If S is algebraic, then the primitive cohomology with respect to a polarization $H_p^2(S, \mathbb{Q})$ and $T(S) = \text{NS}(S)^{\perp}$ are polarized Hodge structures of primitive K3 type.

More generally, if X is an irreducible symplectic variety, then by [Be], the group $H^2(X, \mathbb{Q})$ can be endowed with a quadratic form q which makes the triple $(H^2(X, \mathbb{Q}), h, q)$ into a Hodge structure of K3 type. If X is projective, then the transcendental lattice of X , which is defined as the transcendental lattice of this Hodge structure of K3 type, is a polarized Hodge structure of primitive K3 type. The same is true for the primitive cohomology $H_p^2(X, \mathbb{Q})$ with respect to a polarization.

Let W be an algebraic Hodge structure of K3 type. Then the transcendental lattice $T(W)$ is an irreducible Hodge structure, i.e. $T(W)$ is an irreducible SMT(h')-representation. By Schur's Lemma we get

$$\begin{aligned} \text{End}_{\text{Hdg}}(W, h) &\simeq \text{End}_{\text{Hdg}}(\text{NS}(W)) \oplus \text{End}_{\text{Hdg}}(T(W)) \\ &\simeq \text{End}_{\mathbb{Q}}(\text{NS}(W)) \oplus \text{End}_{\text{Hdg}}(T(W)). \end{aligned} \quad (1.6)$$

In the beginning of this subsection we had identified the space of interesting Hodge classes on the self-product of a K3 surface S with $\text{End}_{\text{Hdg}}(H^2(S, \mathbb{Q}))$. The above decomposition (1.6) tells us

$$B(H^4(S \times S, \mathbb{Q})) \simeq \text{End}_{\mathbb{Q}}(\text{NS}(S)) \oplus \text{End}_{\text{Hdg}}(T(S)) \oplus (\text{algebraic classes}).$$

By the Lefschetz theorem on (1,1) classes the first summand is generated by algebraic classes. Therefore we are led to study in more detail the second one.

1.1.3 Endomorphisms of T

Assume that (T, h, q) is an irreducible, polarized Hodge structure of primitive K3 type. Then, by Schur's lemma, $E := \text{End}_{\text{Hdg}}(T)$ is a division algebra. Zarhin [Z] studied this algebra in detail. We quickly review his results:

The algebra E has a natural involution $'$ which sends an endomorphism φ to his adjoint with respect to q . Moreover, E comes with a natural map

$$\epsilon : \begin{cases} E \rightarrow \mathbb{C} \\ \varphi \mapsto \text{eigenvalue of } \varphi_{\mathbb{C}} \text{ on } T^{2,0} . \end{cases} \quad (1.7)$$

Theorem 1.1.3.1 (Zarhin, see [Z]). (i) *The map ϵ is an embedding of fields. In particular, E is an algebraic number field.*

(ii) *Denote by E_0 the invariant part of E under $'$. Then E_0 is a totally real number field and either*

(a) *$E = E_0$ (we say that T has real multiplication) or*

(b) *E is a purely imaginary quadratic extension of E_0 and $' : E \rightarrow E$ is the restriction of complex conjugation to $E \subset \mathbb{C}$ (T has complex multiplication).*

1.1.4 Mukai's result and K3 surfaces with CM

We now turn back our attention to the study of the Hodge conjecture for the self-product of a K3 surface S . We have seen in 1.1.2 that the most interesting Hodge classes on $S \times S$ can be interpreted as elements of $\text{End}_{\text{Hdg}}(T(S))$.

More generally, interesting Hodge classes on a product of two K3 surfaces S_1, S_2 can be interpreted as morphisms of Hodge structures $\varphi : T(S_1) \rightarrow T(S_2)$.

Mukai uses his theory of moduli spaces of sheaves to produce cycles on products of K3 surfaces. In [Mu2] he announced the following theorem which gives a positive answer to a conjecture by Shafarevich [Sha].

Theorem 1.1.4.1 (Mukai). *Let S_1, S_2 be two algebraic K3 surfaces and let $\varphi : T(S_1) \simeq T(S_2)$ be a Hodge isometry (i.e. an isomorphism of Hodge*

structures which respects the intersection products). Then φ is induced by an algebraic cycle on $S_1 \times S_2$.

The theorem has been proved in the case $\rho(S_i) \geq 11$ in [Mu1], later this has been improved by Nikulin [N] to $\rho(S_i) \geq 5$. However, for the general case no proof has been published yet.

If $S_1 = S_2 = S$ we can think of φ as an element of the algebraic number field $E = \text{End}_{\text{Hdg}}(T(S))$ (which is embedded via ϵ in \mathbb{C}). Since Zarhin's Theorem 1.1.3.1 identifies adjunction with respect to the intersection form with complex conjugation, φ is an isometry if and only if $\varphi\bar{\varphi} = 1$. Combined with the fact that CM fields are generated as \mathbb{Q} -vector spaces by elements of norm 1 (see [Bo]), Mukai's theorem has the following nice consequence.

Corollary 1.1.4.2. *If S is an algebraic K3 surface such that $E = \text{End}_{\text{Hdg}}(T(S))$ is a CM field, then the Hodge conjecture is true for $S \times S$.*

This had been noticed first by Morrison in [Mo], see also [RM]. As a complete proof of Theorem 1.1.4.1 and of Corollary 1.1.4.2 is not available for the time being, we will usually avoid them or at least point out which results really rely upon them.

To study the Hodge conjecture for self-products of a K3 surface S , the interesting open case are endomorphisms of the Hodge structure $T(S)$ which act by real eigenvalues on $H^{2,0}(S)$. For this reason, in the rest of this section we focus on polarized Hodge structures of primitive K3 type with an action by a totally real number field.

1.1.5 Splitting of T over extension fields

(For this and the next subsection see [vG4], 2.4 and 2.5) Let (T, h, q) be an irreducible, polarized Hodge structure of primitive K3 type and assume that $F \subset \text{End}_{\text{Hdg}}(T)$ is a totally real number field. Note that by Theorem 1.1.3.1, F is automatically contained in the ' \prime -invariant part of $\text{End}_{\text{Hdg}}(T)$, i.e. in the subfield of q -self-adjoint endomorphisms.

By the theorem of the primitive element, there exists $\alpha \in F$ such that $F = \mathbb{Q}(\alpha)$. Let $d = [F : \mathbb{Q}]$. Let P be the minimal polynomial of α over \mathbb{Q} , denote by \tilde{F} the splitting field of P in \mathbb{R} . Let $G = \text{Gal}(\tilde{F}/\mathbb{Q})$ and $H = \text{Gal}(\tilde{F}/F)$. Choose $\sigma_1 = \text{id}, \sigma_2, \dots, \sigma_d \in G$ such that

$$G = \sigma_1 H \sqcup \dots \sqcup \sigma_d H.$$

Note that each coset $\sigma_i H$ induces a well-defined embedding $F \hookrightarrow \tilde{F}$. In $\tilde{F}[X]$ we get

$$P(X) = \prod_{i=1}^d (X - \sigma_i(\alpha))$$

and consequently

$$\begin{aligned} F \otimes_{\mathbb{Q}} \tilde{F} &= \mathbb{Q}[X]/(P) \otimes_{\mathbb{Q}} \tilde{F} \\ &\simeq \bigoplus_{i=1}^d \tilde{F}[X]/(X - \sigma_i(\alpha)) \\ &\simeq \bigoplus_{i=1}^d F_{\sigma_i}. \end{aligned}$$

The symbol F_{σ_i} stands for the field \tilde{F} , the index σ_i keeps track of the fact that the F -linear extension of $F \subset \text{End}_{\mathbb{Q}}(F)$ acts on F_{σ_i} via $e(x) = \sigma_i(e) \cdot x$. See Section 2.1.5 for another interpretation of F_{σ_i} .

In the same way, since T is a finite-dimensional F -vector space we get a decomposition

$$T_{\tilde{F}} = T \otimes_{\mathbb{Q}} \tilde{F} = \bigoplus_{i=1}^d T_{\sigma_i}.$$

This is the decomposition of $T_{\tilde{F}}$ into eigenspaces of the \tilde{F} -linear extension of the F -action on T , T_{σ_i} being the eigenspace of $e_{\tilde{F}}$ to the eigenvalue $\sigma_i(e)$ for $e \in F$. Since each $e \in F$ is q -self-adjoint (that is $e' = e$), the decomposition is orthogonal. Let $q_{\tilde{F}}$ be the \tilde{F} -bilinear extension of q to $T_{\tilde{F}} \times T_{\tilde{F}}$. Using the notation

$$T_i := T_{\sigma_i} \text{ and } q_i = (q_{\tilde{F}})|_{T_i \times T_i},$$

we have an orthogonal decomposition

$$(T_{\tilde{F}}, q_{\tilde{F}}) = \bigoplus_{i=1}^d (T_i, q_i). \quad (1.8)$$

1.1.6 Galois action on $T_{\tilde{F}}$

Letting G act in the natural way on \tilde{F} , we get a (only \mathbb{Q} -linear) Galois action on $T_{\tilde{F}} = T \otimes_{\mathbb{Q}} \tilde{F}$. Under this action, for $\tau \in G$ we have

$$\tau T_{\sigma_i} = T_{\tau\sigma_i}. \quad (1.9)$$

This is because the Galois action commutes with the \tilde{F} -linear extension of any endomorphism $e \in F \subset \text{End}_{\mathbb{Q}}(T)$ the latter being defined over \mathbb{Q} and because for $t_i \in T_{\sigma_i}$ and $e \in F$

$$e_{\tilde{F}}(\tau(t_i)) = \tau(e_{\tilde{F}}(t_i)) = \tau(\sigma_i(e)t_i) = \tau(\sigma_i(e))\tau(t_i) = (\tau\sigma_i(e))\tau(t_i),$$

which means that τ permutes the eigenspaces of $e_{\tilde{F}}$ precisely in the way we claimed. Define a homomorphism

$$\gamma : \begin{cases} G \rightarrow \mathfrak{S}_d \\ \tau \mapsto \{i \mapsto \tau(i) \text{ where } (\tau\sigma_i)H = \sigma_{\tau(i)}H\} . \end{cases} \quad (1.10)$$

(This describes the action of G on G/H). With this notation, (1.9) reads

$$\tau T_i = T_{\tau(i)}. \quad (1.11)$$

Interpret T as a subspace of $T_{\tilde{F}}$ via the natural inclusion $T \hookrightarrow T_{\tilde{F}}$, $t \mapsto t \otimes 1$. Denote by π_i the projection to T_i . For $t \in T$ and $\tau \in G$ we have $t = \tau(t)$. Write $t_i := \pi_i(t \otimes 1)$, then $t = \sum_i t_i$. Using (1.11) we see that

$$t_{\sigma_i} = \sigma_i(t_i). \quad (1.12)$$

It follows that

$$\iota_i : \begin{cases} T \rightarrow T_i \\ t \mapsto \pi_i(t \otimes 1) \end{cases} \quad (1.13)$$

is an injective map of F -vector spaces (F acting on T_i via $\sigma_i : F \hookrightarrow \tilde{F}$). Equation (1.12) can be rephrased as

$$\iota_i = \sigma_i \circ \iota_1. \quad (1.14)$$

Since q is defined over \mathbb{Q} , we have for $t \in T_{\tilde{F}}$ and $\tau \in G$

$$q_{\tilde{F}}(\tau t) = \tau q_{\tilde{F}}(t).$$

This implies that for $t \in T$

$$q_i(\iota_i(t)) = \sigma_i q_1(\iota_1(t)). \quad (1.15)$$

1.1.7 Weil restriction

Let L/K be a finite extension of fields, let X be a quasi-projective variety over L . Consider the functor

$$\begin{aligned} \text{res}_{L/K} : (K\text{-varieties})^{\text{op}} &\rightarrow (\text{Sets}) \\ S &\mapsto \text{Hom}_L(S \times_K L, X). \end{aligned}$$

This functor is representable by a quasi-projective K -variety ([BLR], Thm. 7.6.4) which we will denote by $\text{Res}_{L/K}(X)$ and which is called the *Weil restriction* of X to K .

1.1.8 The special Mumford–Tate group of T

Zarhin also computes the special Mumford–Tate group of an irreducible polarized Hodge structure of primitive K3 type. His description distinguishes the cases (a) and (b) in Theorem 1.1.3.1. Since we are more interested in the case of real multiplication we now assume that $E := \text{End}_{\text{Hdg}}(T)$ is a totally real number field. We use the notations of subsections 1.1.5 and 1.1.6 in the special case $F = E = \text{End}_{\text{Hdg}}(T)$.

Denote by Q the restriction of q_1 to $T \subset T_1$ (use the inclusion ι_1 of (1.13)). This is an E -valued (since H -invariant), non-degenerate, symmetric bilinear form on the E -vector space T . Denote by $\text{SO}(Q)$ the E -linear algebraic group of Q -orthogonal, E -linear transformations of T with determinant 1.

Theorem 1.1.8.1 (Zarhin, see [Z], see also [vG4], 2.8). *The special Mumford–Tate group of the Hodge structure (T, h, q) with real multiplication by E is*

$$\text{SMT}(T) = \text{Res}_{E/\mathbb{Q}}(\text{SO}(Q)).$$

Its representation on T is the natural one, where we regard T as a \mathbb{Q} -vector space and use that any E -linear endomorphism of T is in particular \mathbb{Q} -linear. After base change to \tilde{E}

$$\text{SMT}(T)_{\tilde{E}} = \prod_i \text{SO}(T_i, (q_i)),$$

its representation on $T_{\tilde{E}} = \bigoplus_i T_i$ is the product of the standard representations.

Corollary 1.1.8.2 (Van Geemen, see [vG4], Lemma 3.2). *If E is a totally real number field, then $\dim_E T \geq 3$.*

Proof. Let $m = \dim_E T$, let $d = [E : \mathbb{Q}]$. Then $dm = \dim_{\mathbb{Q}} T$ and (1.8) tells us that T splits over \tilde{E} in d summands. By (1.11), these spaces are permuted by the Galois group. Thus $\dim_{\tilde{E}} T_1 = m$.

Now, the case $m = 1$ is excluded by the observation that $T^{2,0} \oplus T^{0,2}$ is contained in $T_{1,\mathbb{C}}$ so that this space has at least dimension 2.

The case $m = 2$ is not possible because otherwise we had $\text{SMT}(V)(\mathbb{C}) = \text{SO}(2, \mathbb{C})^d$. The representation of this group on

$$T_{\mathbb{C}} \simeq \bigoplus_{i=1}^d T_{i,\mathbb{C}} \simeq \bigoplus_{i=1}^d \mathbb{C}^2$$

is the product representation. On the other hand,

$$\text{End}_{\text{Hdg}}(T)_{\mathbb{C}} = \text{End}_{\mathbb{C}}(T_{\mathbb{C}})^{\text{SMT}(T)(\mathbb{C})} = \bigoplus_i \text{End}_{\mathbb{C}}(T_{i,\mathbb{C}})^{\text{SO}(2,\mathbb{C})}.$$

Now the matrices in $\text{Mat}_2(\mathbb{C})$ which commute to $\text{SO}(2, \mathbb{C})$ are precisely the diagonal matrices, because $\text{SO}(2, \mathbb{C}) \simeq \mathbb{C}^*$ and the standard representation of $\text{SO}(2, \mathbb{C})$ on \mathbb{C}^2 is equivalent to the representation of \mathbb{C}^* given by

$$z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

It follows that $\text{End}_{\mathbb{C}}(T_{\mathbb{C}})^{\text{SMT}(T)(\mathbb{C})} = (\mathbb{C}^2)^d$. But this is a contradiction because $\text{End}_{\text{Hdg}}(T)_{\mathbb{C}} = E \otimes_{\mathbb{Q}} \mathbb{C}$ is of dimension d over \mathbb{C} . \square

1.2 The variational approach

In this section we deduce a lemma which describes the Hodge locus of an endomorphism of a Hodge structure of K3 type in the period domain. This locus turns out to be the intersection of the period domain with the projectivization of the eigenspace which contains the period. We apply this to prove Theorem 1 which reduces the Hodge conjecture for self-products of K3 surfaces to Grothendieck's invariant cycle conjecture. Finally, we discuss how real and complex multiplication behave with twistor deformations.

1.2.1 The Hodge locus of an endomorphism

Variations of Hodge structures and Hodge loci. Recall that a (rational) variation of Hodge structures of weight k (VHS) on a complex manifold B consists of a local system of \mathbb{Q} -vector spaces \mathbb{V} of fibre V and a decreasing filtration $(F^l\mathcal{V})_{0 \leq l \leq k}$ of the holomorphic vector bundle $\mathcal{V} := \mathbb{V} \otimes_{\mathbb{Q}} \mathcal{O}_B$ satisfying fibrewise condition (1.3) and satisfying the transversality condition $\nabla(F^l\mathcal{V}) \subset F^{l-1}\mathcal{V} \otimes \Omega_B^1$. Here, ∇ is the Gauss-Manin connection on \mathcal{V} . Thus, for all $b \in B$ we get a Hodge structure of weight k on V and these vary holomorphically.

Denote by $|F^l\mathcal{V}|$ the total space of the vector bundle $F^l\mathcal{V}$.

Definition 1.2.1.1. The locus of Hodge classes of the VHS $(\mathbb{V}, F^l\mathcal{V})$ is the analytic set

$$Z := \{(\alpha, b) \in |F^k\mathcal{V}| \mid \alpha \in F^k\mathcal{V}(b) \cap \mathbb{V}(b)\}.$$

Given a class $\alpha \in F^k\mathcal{V}(b) \cap \mathbb{V}(b)$, define Z_α to be the connected component of Z through (α, b) . The Hodge locus of α is the image of Z_α under the projection $|F^k\mathcal{V}| \rightarrow B$

The period domain for Hodge structures of K3 type. Let (W, q) be a quadratic vector space of dimension r , assume that the signature of q is $(3+, (r-3)-)$. The period domain

$$\Omega := \{[\sigma] \in \mathbb{P}(W_{\mathbb{C}}) \mid q(\sigma) = 0, q(\sigma + \bar{\sigma}) > 0\}$$

parametrizes Hodge structures of K3 type on (W, q) .

Here to $[\sigma] \in \Omega$ we associate the Hodge structure given by

$$h_\sigma : \mathrm{U}(1) \rightarrow \mathrm{GL}(W)_{\mathbb{R}}$$

where via h_σ a complex number $z \in \mathrm{U}(1)(\mathbb{R})$ acts on $\langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma) \rangle_{\mathbb{R}}$ by the matrix $\frac{1}{2} \begin{pmatrix} z^2 + \bar{z}^2 & iz^2 - i\bar{z}^2 \\ iz^2 - i\bar{z}^2 & -z^2 - \bar{z}^2 \end{pmatrix}$ and as the identity on $\langle \mathrm{Re}(\sigma), \mathrm{Im}(\sigma) \rangle_{\mathbb{R}}^{\perp}$. Then

$$W^{2,0} = \mathbb{C}\sigma, \quad W^{0,2} = \mathbb{C}\bar{\sigma} \quad \text{and} \quad W^{1,1} = \langle \sigma, \bar{\sigma} \rangle_{\mathbb{C}}^{\perp}.$$

The line $[\sigma] \in \mathbb{P}(W_{\mathbb{C}})$ is called the *period of the Hodge structure* (W, h_σ) . For more details on the period domain see [Hu].

On Ω we have a natural VHS of weight 2. To see this, note first that the quadratic form q induces a vector bundle homomorphism $\beta : W \otimes_{\mathbb{Q}} \mathcal{O}_{\Omega} \rightarrow \mathcal{O}_{\Omega}(1)$ which is defined by composing the isomorphism $W \simeq W^*$ given by q with the tautological surjection $W^* \otimes_{\mathbb{Q}} \mathcal{O}_{\Omega} \rightarrow \mathcal{O}_{\Omega}(1)$. By the definition of Ω , the tautological subbundle $\mathcal{O}_{\Omega}(-1) \subset W \otimes \mathcal{O}_{\Omega}$ is contained in $\ker(\beta)$.

Now the VHS on Ω is given by the constant sheaf W_{Ω} of fibre W and the filtration of $\mathcal{W} = W_{\Omega} \otimes \mathcal{O}_{\Omega}$ is given by

$$F^0\mathcal{W} := \mathcal{W}, \quad F^1\mathcal{W} = \ker(\beta) \quad \text{and} \quad F^2\mathcal{W} = \mathcal{O}_{\Omega}(-1) \subset \mathcal{W}$$

Clearly the $F^i\mathcal{W}$ satisfy pointwise condition (1.3) and the transversality condition follows from the fact that β has constant coefficients.

The Hodge locus of an endomorphism. Let $[\sigma_0] \in \Omega$, let $\varphi \in \text{End}_{\text{Hdg}}(W, h_{\sigma_0})$. We are interested in the Hodge locus Ω_{φ} of $(\varphi, [\sigma_0])$, seen as a section of the VHS $W_{\Omega}^* \otimes W_{\Omega}$, i.e. in the connected component passing through $[\sigma_0]$ of

$$\tilde{\Omega}_{\varphi} := \{[\sigma] \in \Omega \mid \varphi_{\mathbb{R}} \circ h_{\sigma} = h_{\sigma} \circ \varphi_{\mathbb{R}}\}.$$

The quadratic form q induces an involution $' : \text{End}_{\text{Hdg}}(W, h_{\sigma}) \rightarrow \text{End}_{\text{Hdg}}(W, h_{\sigma})$, determined by the equation $q(ev, w) = q(v, e'w)$. Then the result is

Lemma 1.2.1.2. *Let $\lambda \in \mathbb{C}$ be the eigenvalue of φ to the eigenvector σ_0 . Denote by $W_{\lambda} \subset W_{\mathbb{C}}$ the eigenspace to λ . Then*

$$\Omega_{\varphi} \subset \mathbb{P}(W_{\lambda}) \cap \Omega.$$

If $\varphi' = \varphi$, then the inclusion is an equality.

Proof. Consider the Hodge decomposition

$$W_{\mathbb{C}} = W^{2,0} \oplus W^{0,2} \oplus W^{1,1} = \mathbb{C}\sigma_0 \oplus \mathbb{C}\bar{\sigma}_0 \oplus W^{1,1}$$

corresponding to the Hodge structure h_{σ_0} . Let $[\sigma] \in \tilde{\Omega}_{\varphi}$, write $\sigma = a\sigma_0 + b\bar{\sigma}_0 + \sigma^{1,1}$. Then

$$\varphi(\sigma) = \lambda a\sigma_0 + \bar{\lambda} b\bar{\sigma}_0 + \varphi(\sigma^{1,1})$$

and $\varphi(\sigma^{1,1}) \in W^{1,1}$. Since σ is an eigenvector of φ , it must be either an eigenvector to λ or $\bar{\lambda}$ or $\sigma \in W^{1,1}$. But $\langle \text{Re}(\sigma), \text{Im}(\sigma) \rangle_{\mathbb{R}}$ is a positive two-plane and the signature of the restriction of $q_{\mathbb{R}}$ to $W_{\mathbb{R}}^{1,1}$ is $(1+, (\dim_{\mathbb{Q}}(W) - 2)-)$. This implies that σ cannot be contained in $W^{1,1}$. Thus $\sigma \in \mathbb{P}(W_{\lambda}) \cup \mathbb{P}(\bar{W}_{\lambda})$. Since $\mathbb{P}(W_{\lambda}) \cap \mathbb{P}(\bar{W}_{\lambda}) = \emptyset$, we get $\Omega_{\varphi} \subset \mathbb{P}(W_{\lambda})$.

Conversely, assume that $\varphi' = \varphi$, let $[\sigma] \in \mathbb{P}(W_{\lambda}) \cap \Omega$. The hypothesis implies that $\varphi(\bar{\sigma}) = \bar{\lambda}\bar{\sigma}$ and that $\varphi(\langle \sigma, \bar{\sigma} \rangle^{\perp}) \subset \langle \sigma, \bar{\sigma} \rangle^{\perp}$. Therefore, φ respects the Hodge structure h_{σ} . \square

1.2.2 Proof and discussion of Theorem 1

Theorem 1. *Let S be a K3 surface with real multiplication by a totally real number field $E = \text{End}_{\text{Hdg}}(T(S))$. Let $\varphi \in E$. Then there exist a smooth, projective morphism of smooth, quasi-projective, connected varieties $\pi : \mathcal{X} \rightarrow B$, a base point $0 \in B$ with fiber $\mathcal{X}_0 = \pi^{-1}(0) = S$ and a dense subset (with respect to the classical topology) $\Sigma \subset B$ with the following properties:*

- (i) φ is monodromy-equivariant,
- (ii) for each $s \in \Sigma$ the homomorphism $\varphi_s \in \text{End}_{\mathbb{Q}}(H^2(\mathcal{X}_s, \mathbb{Q}))$, obtained by parallel transport of φ , is algebraic.

Proof. The plan of the proof is to use Lemma 1.2.1.2 combined with Corollary 1.1.8.2 to show that the Hodge locus of φ is of positive dimension in some Hilbert scheme parameterizing projective deformations of $S \subset \mathbb{P}^N$. Then we apply again Corollary 1.1.8.2 to show that this Hodge locus contains a dense subset parameterizing K3 surfaces with CM. Together with Mukai's Corollary 1.1.4.2 this will finish the proof. Let's work this now out in detail.

Fix an ample line bundle \mathcal{L} on S , let $\alpha = c_1(\mathcal{L})$. Let $(\Gamma, q) := (H^2(S, \mathbb{Z}), \cup)$, let Ω be the period domain

$$\Omega := \{[\sigma] \in \mathbb{P}(\Gamma_{\mathbb{C}}) \mid q(\sigma) = 0 \text{ and } q(\sigma + \bar{\sigma}) > 0\}.$$

For $[\tau] \in \Omega$, define

$$\begin{aligned} \Delta_{\tau} &:= \{\delta \in \Gamma \mid q(\delta, \tau) = 0 \text{ and } q(\delta) = -2\}, \\ \Omega_{\alpha} &:= \{[\sigma] \in \Omega \mid q(\alpha, \sigma) = 0\} \text{ and} \\ \Omega_{\alpha}^0 &:= \{[\sigma] \in \Omega_{\alpha} \mid \forall \delta \in \Delta_{\sigma} : q(\alpha, \delta) \neq 0\}. \end{aligned}$$

Then Ω_{α} parametrizes those Hodge structures where α is of type $(1, 1)$ and Ω_{α}^0 those for which α lies on no wall of the Weyl chamber structure of the positive cone (see [K3], Exposé VII). Moreover, Ω_{α}^0 is an open subset of Ω_{α} .

Let $x = [H^{2,0}(S)] \in \Omega$ be the point corresponding to the period of S . The germ (Ω_{α}^0, x) of Ω_{α}^0 at x is a germ of a universal deformation of the polarized surface (S, \mathcal{L}) . Since Ω_{α}^0 is smooth, this germ is irreducible.

Recall that on a K3 surface S , the third power of an ample line bundle is very ample. Let $N + 1 = h^0(S, \mathcal{L}^{\otimes 3})$ and choose a basis of $H^0(S, \mathcal{L}^{\otimes 3})$. This yields an embedding $S \subset \mathbb{P}^N$. We choose an irreducible component H of the open subset of the Hilbert scheme parametrizing smooth deformations of the embedding $S \subset \mathbb{P}^N$ which has the property: The germ of H at $b = [S \subset \mathbb{P}^N]$ surjects onto (Ω_{α}^0, x) via the period map induced by the universal family over H . We will show that $H^1(S, \mathcal{N}_{S|\mathbb{P}^N}) = 0$. This implies that H is smooth at b (see e.g. [FG, Cor. 6.4.11]) and the same argument applied to an arbitrary point of H shows that H is smooth everywhere. To see that

$H^1(S, \mathcal{N}_{S|\mathbb{P}^N}) = 0$, consider the exact sequence coming from the normal bundle sequence

$$H^1(S, \mathcal{T}_S) \xrightarrow{\beta} H^1(S, \mathcal{T}_{\mathbb{P}^N|S}) \longrightarrow H^1(S, \mathcal{N}_{S|\mathbb{P}^N}) \longrightarrow H^2(S, \mathcal{T}_S) = 0.$$

We have to show that β is surjective. Consider the following piece of the long exact cohomology sequence induced by the restriction of the Euler sequence to S

$$H^1(S, (\mathcal{L}^{\otimes 3})^{\oplus N+1}) \rightarrow H^1(S, \mathcal{T}_{\mathbb{P}^N|S}) \rightarrow H^2(S, \mathcal{O}_S) \rightarrow H^2(S, (\mathcal{L}^{\otimes 3})^{\oplus N+1}).$$

Since by the Kodaira vanishing theorem $H^1(S, \mathcal{L}^{\otimes 3}) = H^2(S, \mathcal{L}^{\otimes 3}) = 0$ and since $H^2(S, \mathcal{O}_S) \simeq \mathbb{C}$, this shows that $H^1(S, \mathcal{T}_{\mathbb{P}^N|S}) \simeq \mathbb{C}$. Now, the Serre dual of the homomorphism β is the map

$$\beta^* : \mathbb{C} \simeq H^1(S, \Omega_{\mathbb{P}^N|S}) \rightarrow H^1(S, \Omega_S)$$

which is given by restricting the class of the Fubini–Study metric from \mathbb{P}^N to S . Clearly, this map is injective, whence β is surjective.

Thus, we have a smooth, projective morphism of smooth, quasi-projective \mathbb{C} -varieties $\pi' : \mathcal{Y} \rightarrow H$ and a point $b \in H$ with $\mathcal{Y}_b \simeq S$ such that via the period map an open neighborhood of b is mapped surjectively onto an open neighborhood of y in Ω_α^0 .

Over H we have the VHS $\mathbb{V} = R^2\pi'_*\mathbb{Q}_{\mathcal{Y}} \otimes R^2\pi'_*\mathbb{Q}_{\mathcal{Y}}$, let $\mathcal{V} := \mathbb{V} \otimes \mathcal{O}_H$. Let $B' \subset |F^2\mathcal{V}|$ be the connected component of the locus of Hodge classes of \mathbb{V} passing through (φ, b) , where $F^2\mathcal{V}$ is the second step of the Hodge filtration of \mathcal{V} and we interpret φ as a class in $(\mathbb{V} \otimes \mathbb{V})(b)$ by identifying $H^2(S, \mathbb{Q})$ with $H^2(S, \mathbb{Q})^*$ via Poincaré duality. By the famous result of Cattani, Deligne and Kaplan [CDK], B' is a closed algebraic subset of $|F^2\mathcal{V}|$. Resolving the singularities of B' we obtain a smooth, quasi-projective variety B and by pullback a smooth, projective morphism $\pi : \mathcal{X} \rightarrow B$. Pick a point 0 in the fibre of $\sigma : B \rightarrow H$ over b , then $\mathcal{X}_0 \simeq S$.

Tautologically, the bundle $F^2(R^2\pi_*\mathbb{Q}_{\mathcal{X}} \otimes R^2\pi_*\mathbb{Q}_{\mathcal{X}} \otimes \mathcal{O}_B) = \sigma^*F^2\mathcal{V}$ has a holomorphic section $\tilde{\varphi}$ such that $\tilde{\varphi}(0) = \varphi$ and for all $t \in B$

$$\tilde{\varphi}(t) \in (\sigma^*F^2\mathcal{V}(t)) \cap (R^2\pi_*\mathbb{Q}_{\mathcal{X}} \otimes R^2\pi_*\mathbb{Q}_{\mathcal{X}}(t)).$$

This implies that $\tilde{\varphi}$ is a global section of $R^2\pi_*\mathbb{Q}_{\mathcal{X}} \otimes R^2\pi_*\mathbb{Q}_{\mathcal{X}}$, i.e. that $\varphi \in \text{End}_{\text{Hdg}} T(\mathcal{X}_0)$ is monodromy-equivariant. Thus, π satisfies condition (i).

As for (ii), we show first that B has positive dimension. By Lemma 1.2.1.2 we are led to study the eigenspace $T_\lambda \subset T(S)_{\mathbb{C}}$ of φ which contains σ_S . The same argument as in the proof of Corollary 1.1.8.2 shows that $\dim_{\mathbb{C}} T_\lambda = \dim_F T(S)$ where $F \subset \text{End}_{\text{Hdg}}(T(S)) =: E$ is the subfield generated by φ . Then by Corollary 1.1.8.2,

$$\dim \mathbb{P}(T_\lambda) \cap \Omega_\alpha = \dim_F(T(S)) - 2 \geq \dim_E T(S) - 2 \geq 1$$

(we loose one dimension by projectivization and one dimension by intersecting with the quadric Ω). Consequently, $\mathbb{P}(T_\lambda) \cap \Omega_\alpha^0$ is a positive-dimensional space. Clearly, the germ $(B, 0)$ surjects onto $(\mathbb{P}(T_\lambda) \cap \Omega_\alpha^0, x)$ because (H, b) does so on (Ω_α^0, x) . This implies that B is of positive dimension.

Now, we show that there is a set $\Sigma \subset B$ which is dense and such that the K3 surfaces parametrized by Σ have CM. This will finish the proof of the theorem. Note first that for $t \in B$, the Picard number $\rho(\mathcal{X}_t)$ can only be greater than or equal to $\rho(S)$ because $\text{Pic}(S)$ is orthogonal to T_λ . The space $T(\mathcal{X}_t)$ is an F -vector space, write m_t for its dimension. Then we have

$$22 = m_t d + \rho(\mathcal{X}_t), \quad (1.16)$$

where $d = [F : \mathbb{Q}]$. This formula shows that the Picard number in the family \mathcal{X} behaves in an interesting way: it can only be of the form $\rho(S) + kd$ for $k \geq 0$. By [O, Thm. 1.1], the locus

$$B^+ := \{t \in B \mid \rho(\mathcal{X}_t) > \rho(S)\}$$

is dense in the classical topology in B . Then for $t \in B^+$ one can consider the restriction of the family $\mathcal{X} \rightarrow B$ to the Hodge locus of a set of generators of $\text{Pic}(\mathcal{X}_t)$. This yields a family in which each fibre has Picard number greater than or equal to $\rho(\mathcal{X}_t) > \rho(S)$ and consequently by (1.16) $m_s < m_0$ for s in the base of such a subfamily. Continue this argument inductively to see that

$$\Sigma = \{t \in B \mid m_t < 3\}$$

is dense in B . Now apply Corollary 1.1.8.2 to see that for $t \in \Sigma$ the algebraic number field $\text{End}_{\text{Hdg}}(T(\mathcal{X}_t))$ is a CM field. \square

Remark. Note that in the proof we only use the part of Mukai's theorem for which has a rigorous proof has been written up. Indeed, for $t \in \Sigma$ we have $\dim_{\mathbb{Q}}(T(\mathcal{X}_t)) = m_t d \leq 2d \leq 14$ since $d \leq 7$ (use Corollary 1.1.8.2 and $\dim_{\mathbb{Q}} T(S) = md \leq 21$). Thus $\rho(\mathcal{X}_t) \geq 8$ and we can use Nikulin's improvement of Mukai's theorem (see the remark below Theorem 1.1.4.1).

The interest in the theorem stems from

Grothendieck's invariant cycle conjecture. *Let $\pi : \mathcal{X} \rightarrow B$ be a smooth, projective morphism of smooth, connected, quasi-projective \mathbb{C} -varieties. Let α be a global section of $R^{2k}\pi_*\mathbb{Q}_{\mathcal{X}}$ for some natural number k .*

If there exists a point $b_0 \in B$ such that $\alpha(b_0)$ is algebraic in $H^{2k}(\mathcal{X}_{b_0}, \mathbb{Q})$, then for all points $b \in B$ the class $\alpha(b)$ is algebraic in $H^{2k}(\mathcal{X}_b, \mathbb{Q})$.

Remark. i) Theorem 1 reduces the Hodge conjecture for self-products of K3 surfaces to Grothendieck's invariant cycle conjecture. This reduction

was known before by arguments of André [An1]. They run along the following line: Show first that the Kuga–Satake correspondence is algebraic if Grothendieck’s invariant cycle conjecture holds. This is done by using the density of K3 surfaces of maximal Picard rank in the moduli space. The next step is to use that the invariant cycle conjecture implies the Hodge conjecture for Abelian varieties. Our argument is more direct and the hope was that this could lead to a more attackable approach. See ii).

ii) André has shown in [An4] that the invariant cycle conjecture is true for a family $\pi : \mathcal{X} \rightarrow B$ if the fibers \mathcal{X}_b satisfy the standard conjecture $B(\mathcal{X}_b)$ and if there exists a smooth projectivization $\overline{\mathcal{X}}$ of \mathcal{X} which satisfies the standard conjecture $B(\overline{\mathcal{X}})$ (see Section 4.2.2 for a formulation of this conjecture). The argument is quite tricky: The hypothesis implies by Jannsen’s theorem (see [Ar], Theorem 4.1) that the category of Grothendieck’s homological motives modeled on such products is graded and semi-simple Abelian. Then André shows that there is a submotive $\mathfrak{h}^{2k}(\mathcal{X}_{b_0})^{\pi_1(B, b_0)}$ of $\mathfrak{h}^{2k}(\mathcal{X}_{b_0})$ and a motivic surjection $\mathfrak{h}^{2k}(\overline{\mathcal{X}}) \rightarrow \mathfrak{h}^{2k}(\mathcal{X}_{b_0})^{\pi_1(B, b_0)}$. Next, if $\alpha(b_0) \in H^{2k}(\mathcal{X}_{b_0})$ is algebraic, this class induces a submotive $\mathbb{1} \subset \mathfrak{h}^{2k}(\mathcal{X}_{b_0})^{\pi_1(B, b_0)}$ which then by semi-simplicity and surjectivity comes from a submotive $\mathbb{1} \subset \mathfrak{h}^{2k}(\overline{\mathcal{X}})$. This last submotive induces an algebraic class on $\overline{\mathcal{X}}$ which restricts fibrewise to α .

Therefore, in order to apply Theorem 1 we are “reduced” to show the standard conjecture $B(\overline{\mathcal{X}})$ for a smooth compactification $\overline{\mathcal{X}}$ of the total space of a pencil of self-products of K3 surfaces. Indeed, the fibers of such a pencil satisfy the standard conjecture B since they are products of surfaces (this is an immediate consequence of the Lefschetz theorem on $(1, 1)$ classes).

To make this more concrete, look at the universal quartic hypersurface $\mathcal{Y} \subset \mathbb{P}^3 \times \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$. This variety is a smooth, ample hypersurface in $\mathbb{P}^3 \times \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$ and therefore, it satisfies the standard conjecture $B(\mathcal{Y})$. Consider the morphism $\pi' : \mathcal{Y} \rightarrow \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$ and assume that $x \in \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$ is a point such that $\mathcal{Y}_x = \pi'^{-1}(x)$ is a smooth K3 surface with real multiplication. Then for $\varphi \in \text{End}_{\text{Hdg}}(T(\mathcal{Y}_x)) \setminus \mathbb{Q}$ we can find an irreducible curve $C' \subset \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$ which is contained in the Hodge locus of the class φ and which connects x to a point y for which we know that φ is algebraic on $\mathcal{Y}_y \times \mathcal{Y}_y$. By passing to the normalization \widetilde{C}' of C' and possibly to a finite cover of \widetilde{C}' (in order to make φ monodromy-invariant over the smooth part of the family), we obtain a family $\mathcal{X} \rightarrow C$ such that all but finitely many fibers are smooth quartic K3 surfaces which admit the monodromy-invariant endomorphism of Hodge structures φ on their second cohomology groups. Moreover, \mathcal{X} is still an ample hypersurface, this time in $\mathbb{P}^3 \times C$. If \mathcal{X} was smooth we could conclude that \mathcal{X} satisfies the standard conjecture $B(\mathcal{X})$. However we can show that \mathcal{X} is not smooth. I don’t know whether one can deduce that a resolution of singularities of \mathcal{X} satisfies conjecture B . But even this would not be sufficient for our purposes, what

we are actually interested in, is the standard conjecture B for a resolution of the variety $\mathcal{X} \times_C \mathcal{X}$.

1.2.3 Twistor lines

While in the preceding section we were interested in projective deformations of a K3 surface S we focus now on another distinguished class of deformations, the twistor lines. They come from Hyperkähler structures on K3 surfaces.

In general, we say that a Riemannian metric g on a differentiable manifold M is a *Hyperkähler metric* if there exist complex structures I, J and K on M with $I \circ J = -J \circ I = K$ such that g is a Kähler metric for I, J and K . The quadruple (M, I, J, K, g) is called a *Hyperkähler manifold*.

For $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{S}^2$, we get a complex structure $L_\lambda = \lambda_1 I + \lambda_2 J + \lambda_3 K$ on M . Using the isomorphism $\mathbb{S}^2 \simeq \mathbb{P}^1$, we obtain a holomorphic family $\pi : \mathcal{X} \rightarrow \mathbb{P}^1$ such that $\mathcal{X}_\lambda \simeq (M, L_\lambda)$. This family is called the *twistor line of (M, I) associated with g* . We refer the reader to [GHJ] for more details.

A *trianalytic subvariety* of a Hyperkähler manifold (M, I, J, K, g) is a closed subset $N \subset M$ which is analytic with respect to the complex structures I, J and K . Similarly, a Hermitian connection ∇ on a Hermitian vector bundle \mathcal{E} on M is called *hyperholomorphic* if ∇ is integrable with respect to the complex structures I, J and K . Verbitsky studied these notions for compact Hyperkähler manifolds. He obtained very nice criteria for a subvariety to be trianalytic or for a vector bundle to admit a hyperholomorphic connection.

Theorem 1.2.3.1 (Verbitsky, see [Ve1], Thm. 2.5 and [Ve2], Thm. 3.1). *Let (M, I, J, K, g) be a compact Hyperkähler manifold.*

(i) *Let $N \subset (M, I)$ be a closed, analytic subset of codimension k . Then N is trianalytic if and only if $[N]$ is of type (k, k) with respect to the complex structures J and K .*

(ii) *Let \mathcal{E} be a holomorphic vector bundle on (M, I) . Then \mathcal{E} admits a hyperholomorphic connection if and only if \mathcal{E} is a ω -polystable vector bundle and if $c_1(\mathcal{E})$ and $c_2(\mathcal{E})$ are Hodge classes with respect to J and K .*

Let now $X = (M, I)$ be an irreducible, symplectic variety, that is X is a simply connected, compact Kähler manifold with a nowhere vanishing holomorphic two-form σ_X such that $H^0(X, \Omega_X^2) = \mathbb{C}\sigma_X$ (cf. [Be], [GHJ, Part III]). For each Kähler class $\omega \in \mathcal{K}_X$, there exists a unique Hyperkähler metric g on M with Kähler class ω . Beauville [Be] and Bogomolov introduced a quadratic form q on $H^2(X, \mathbb{Q})$ (actually it is defined on the integral cohomology) which turns $(H^2(X, \mathbb{Q}), q)$ into a Hodge structure of K3 type. Let $\Omega = \{[\sigma] \in \mathbb{P}(H^2(X, \mathbb{C})) \mid q(\sigma) = 0, q(\sigma + \bar{\sigma}) > 0\}$ be the corresponding period domain. Then the period map associated with the twistor line to

$\omega \in \mathcal{K}_X$ induces an isomorphism

$$\mathcal{P} : \mathbb{P}^1 \rightarrow \mathbb{P}(F(\omega)) \cap \Omega, \quad (1.17)$$

where $F(\omega) \subset H^2(X, \mathbb{C})$ is the three-space spanned by $\sigma_X, \bar{\sigma}_X, \omega$.

We assume now that X is projective. Let $T(X)$ be the Beauville–Bogomolov orthogonal complement of the Néron–Severi group of X . The pair $(T(X), q)$ is an irreducible Hodge structure of primitive K3 type, hence Zarhin’s Theorem 1.1.3.1 applies and we distinguish the cases that X has real or complex multiplication.

Examples of irreducible symplectic varieties include K3 surfaces and their Hilbert schemes of points. We are interested in representing Hodge classes on the self-product $S \times S$ of a K3 surface S or on $\text{Hilb}^2(S)$ (see Chapter 3) which correspond to endomorphisms of the transcendental lattice. A possible approach for this problem is to represent such a class on a deformation of $S \times S$ resp. of $\text{Hilb}^2(S)$ by an algebraic cycle Z and to deform Z to an algebraic cycle on $S \times S$ resp. $\text{Hilb}^2(S)$.

In general, it is very difficult to decide whether an algebraic cycle deforms when its ambient variety is deformed. Verbitsky’s criterion shows that the deformation theory of algebraic cycles along twistor is very well behaved. For this reason, the following question arises naturally in our context.

Question 1.2.3.2. Do real or complex multiplication deform along twistor lines?

Let $\omega \in \mathcal{K}_X$, let $\varphi \in \text{End}_{\text{Hdg}}(H^2(X, \mathbb{Q}))$, let $W_\mu \subset H^2(X, \mathbb{C})$ be the eigenspace to $\mu := \epsilon(\varphi)$. Lemma 1.2.1.2 combined with (1.17) tells us that if φ induces via parallel transport an endomorphism of Hodge structures on all fibers of the twistor deformation $\mathcal{X}(\omega) \rightarrow \mathbb{P}^1$, then $F(\omega) \subset W_\mu$. Since $\bar{\sigma}_X \in F(\omega)$ and since $\varphi(\bar{\sigma}_X) = \bar{\mu}\bar{\sigma}_X$, we see that this condition can only be satisfied if $\mu \in \mathbb{R}$. From this we deduce that complex multiplication does not deform along twistor lines. For real multiplication the situation is different:

Proposition 1.2.3.3. *Let $\varphi_t \in \text{End}_{\text{Hdg}}(T(X))$ with $\epsilon(\varphi_t) = \sqrt{d}$ for a rational $d > 0$ which is no square in \mathbb{Q} . Assume that the Picard number $\rho(X)$ is greater than or equal to three.*

Then there exist an endomorphism $\varphi_n : \text{NS}(X) \rightarrow \text{NS}(X)$ and a Kähler class $\omega \in \mathcal{K}_X$ with the property:

For all $x \in \mathbb{P}^1$ the endomorphism $\varphi_x \in \text{End}_{\mathbb{Q}}(H^2(\mathcal{X}(\omega)_x, \mathbb{Q}))$ obtained by parallel transport of $\varphi = \varphi_t + \varphi_n$ respects the Hodge structure on $H^2(\mathcal{X}(\omega)_x, \mathbb{Q})$.

The proposition implies that all projective fibers $\mathcal{X}(\omega)_x$ of the twistor line admit an inclusion $\mathbb{Q}(\sqrt{d}) \subset \text{End}_{\text{Hdg}}(T(\mathcal{X}(\omega)_x))$. If we were able to find a polystable vector bundle \mathcal{E} on one fibre $\mathcal{X}(\omega)_x$ of the twistor line with $\deg_\omega c_1(\mathcal{E}) = 0$ and $c_2(\mathcal{E}) = \varphi(x)$, then Verbitsky’s criterion would imply

that \mathcal{E} is hyperholomorphic. In this way, one could prove that for all $y \in \mathbb{P}^1$ the subfield $\mathbb{Q}(\sqrt{d}) \subset \text{End}_{\text{Hdg}}(T(\mathcal{X}(\omega)_y))$ is generated by algebraic classes.

Proof (Proposition). Using Lemma 1.2.1.2, it is enough to prove the existence of a q -self-adjoint endomorphism $\varphi_n : \text{NS}(X) \rightarrow \text{NS}(X)$ and of a Kähler class $\omega \in \mathcal{K}_X$ with $\varphi_n(\omega) = \sqrt{d}\omega$.

Assume that we have proved the existence of rational Kähler classes ω_1, ω_2 such that $dq(\omega_1) = q(\omega_2)$. Then ω_1, ω_2 are linearly independent and we define

$$\begin{aligned} \varphi_n : \text{NS}(X) &\rightarrow \text{NS}(X) \\ \omega_1 &\mapsto \omega_2 \\ \omega_2 &\mapsto d\omega_1 \\ \alpha &\mapsto 0 \text{ for all } \alpha \in \langle \omega_1, \omega_2 \rangle^\perp \end{aligned}$$

The class $\omega := \sqrt{d}\omega_1 + \omega_2$ is a Kähler class and $\varphi_n(\omega) = \sqrt{d}\omega$. Moreover, φ_n is q -self-adjoint because $q(\varphi_n(\omega_1), \omega_2) = q(\omega_2)$ and $q(\omega_1, \varphi_n(\omega_2)) = dq(\omega_1) = q(\omega_2)$. Thus, φ_n and ω fulfill the requirements of the proposition and we are reduced to prove the existence of $\omega_1, \omega_2 \in \mathcal{K}_X \cap \text{NS}(X)$ with $dq(\omega_1) = q(\omega_2)$.

Let $W := \text{NS}(X) \oplus \text{NS}(X)$, let $r : W \rightarrow \mathbb{Q}$ be the quadratic form defined by $r(\alpha, \beta) := dq(\alpha) - q(\beta)$. We have to show that the intersection of $Q := Z(r) \subset W$ with $\mathcal{K}_X \times \mathcal{K}_X$ is non-empty. It is easily seen that r is a non-degenerate, indefinite quadratic form. Since the dimension n of W is at least 6, by [Se, Cor. I.IV.3.2] there exists $w_0 \in W$ with $r(w_0) = 0$. But this implies that Q is dense in $Q_{\mathbb{R}} := Z(r) \subset W_{\mathbb{R}}$ in the classical topology. Indeed, choose a basis of W such that $w_0 = (1, 0, \dots, 0)$. Then r is of the form $r = \sum_{i,j \geq 2} a_{i,j} X_i X_j + \sum_{i \geq 2} b_i X_i X_1$. Since r is non-degenerate, at least one of the b_i is non-zero. Now we define $f : \mathbb{Q}^{n-1} \setminus Z(\sum_i b_i X_i) \rightarrow Q$ and $f_{\mathbb{R}} : \mathbb{R}^{n-1} \setminus Z(\sum_i b_i X_i) \rightarrow Q_{\mathbb{R}}$ by the formula

$$(x_2, \dots, x_n) \mapsto \left(\frac{\sum_{i,j} a_{i,j} x_i x_j}{\sum_i b_i x_i}, x_2, \dots, x_n \right).$$

Then $\text{im}(f_{\mathbb{R}}) = Q_{\mathbb{R}} \setminus (Z(\sum_i b_i X_i) \cap Q_{\mathbb{R}})$ is dense in $Q_{\mathbb{R}}$ in the classical topology. Here, we use that r is non-degenerate whence the complement of the intersection of a linear subspace of $W_{\mathbb{R}}$ with $Q_{\mathbb{R}}$ is dense in $Q_{\mathbb{R}}$. Clearly $\text{im}(f)$ is dense in $\text{im}(f_{\mathbb{R}})$ which proves that Q is dense in $Q_{\mathbb{R}}$. In more geometric terms, we have used that the existence of $w_0 \in Q$ induces a via projection with center w_0 a birational morphism $Q \rightarrow \mathbb{Q}^{n-1}$.

Now, the intersection $(\mathcal{K}_X \times \mathcal{K}_X) \cap Q_{\mathbb{R}}$ is a non-empty open subset of $Q_{\mathbb{R}}$. It follows, that $(\mathcal{K}_X \times \mathcal{K}_X) \cap Q$ is non-empty. This concludes the proof of the proposition. \square

Chapter 2

The Kuga–Satake correspondence

2.1 Kuga–Satake varieties and real multiplication

In this section, we study the Kuga–Satake variety of a polarized Hodge structure of primitive K3 type with real multiplication. As the main result of the section we show that this Abelian variety is isogenous to a self-product of a certain smaller Abelian variety. This result is a mild improvement of a result of van Geemen. It allows us to calculate the endomorphism algebra of the Kuga–Satake variety. This will be made explicit in a concrete example.

We start by recalling Clifford algebras and the definition of the Kuga–Satake variety. Next, we review the corestriction of algebras, a tool from the theory of central simple algebras which enters the game in the decomposition result. After stating the precise version of this theorem we are concerned with the somewhat lengthy and technical proof. In the last subsection we discuss examples.

2.1.1 Clifford algebras

Let K be a field and let (T, q) be a quadratic K -vector space. The Clifford algebra $C(q)$ is defined as the K -algebra

$$\left(\bigoplus_{i \geq 0} T^{\otimes i} \right) / I(q),$$

where $I(q)$ is the two-sided ideal generated by elements of the form $t \otimes t - q(t)$ for $t \in T$. Choose an orthogonal basis e_1, \dots, e_n of T . Then as a K -vector space

$$C(q) = \bigoplus_{\mathbf{a} \in \{0,1\}^n} K e_1^{a_1} \cdots e_n^{a_n},$$

and

$$e_i^2 = q(e_i) \text{ and } e_i \cdot e_j = -e_j \cdot e_i \text{ for } i \neq j.$$

There is a natural K -linear involution $\iota : C(q) \rightarrow C(q)$ which is induced by $e_1^{a_1} \cdot \dots \cdot e_n^{a_n} \mapsto e_n^{a_n} \cdot \dots \cdot e_1^{a_1}$.

Let $n = \dim_K T$. Then, the dimension of $C(q)$ as a K -vector space is 2^n . There is a natural decomposition

$$C(q) = C^0(q) \oplus C^1(q),$$

where

$$C^i(q) = \bigoplus_{\mathbf{a} \in \{0,1\}^n: \sum_j a_j \equiv i \pmod{2}} K e_1^{a_1} \cdot \dots \cdot e_n^{a_n}.$$

Note that $C^0(q)$ is a subalgebra, the so-called *even Clifford algebra*, which is stable under ι .

2.1.2 Spin group and spin representation

We keep the notations of the previous subsection. The spin group $\text{Spin}(q)$ of q is an algebraic group over K which has the property that for any field extension L/K

$$\text{Spin}(q)(L) = \{v \in (C^0(q_L))^* \mid vT_Lv^{-1} \subset T_L \text{ and } v \cdot \iota(v) = 1\}.$$

This group comes with two natural representations: By definition $\text{Spin}(q)(L)$ acts on T_L via $(v, t) \mapsto vt v^{-1}$. It turns out that this action is orthogonal and with determinant 1 such that there is a homomorphism of algebraic groups

$$\rho : \text{Spin}(q) \rightarrow \text{SO}(q).$$

The kernel of $\rho(L)$ consists of $\{\pm 1\}$ and $\rho(L)$ is surjective for algebraically closed fields L .

The spin representation is the natural (faithful) representation

$$\text{Spin}(q) \hookrightarrow \text{GL}(C^0(q))$$

where $v \in \text{Spin}(q)$ acts on $C^0(q)$ by left multiplication.

2.1.3 Graded tensor product

The graded tensor product $A \widehat{\otimes}_K B$ of two K -superalgebras $A = A^0 \oplus A^1$ and $B = B^0 \oplus B^1$ is defined as the K -vector space $A \otimes B$ with K -algebra structure defined by $(a \otimes b) \cdot (a' \otimes b') = \epsilon(aa') \otimes (bb')$, where $\epsilon = -1$ if $a' \in A^1$ and $b \in B^1$ and $\epsilon = 1$ else.

If $(T, q) = \bigoplus_i^d (T_i, q_i)$, then one gets a natural isomorphism

$$C(q) \simeq C(q_1) \widehat{\otimes} \dots \widehat{\otimes} C(q_d). \quad (2.1)$$

2.1.4 Kuga–Satake varieties

Let (T, q, ψ) be a polarized \mathbb{Q} -Hodge structure of primitive K3 type. Kuga and Satake [KS] found a way to associate to this a polarizable \mathbb{Q} -Hodge structure of weight one (V, h_s) , in other words an isogeny class of Abelian varieties, together with an inclusion of Hodge structures

$$T \subset V \otimes V.$$

The definition of the Kuga–Satake Hodge structure. Set $V := C^0(q)$ and define a weight one Hodge structure on V in the following way: Choose $f_1, f_2 \in (T^{2,0} \oplus T^{0,2})_{\mathbb{R}}$ such that $\mathbb{C}(f_1 + if_2) = T^{2,0}$ and $q(f_i, f_j) = \delta_{i,j}$ (recall that $q|_{(T^{2,0} \oplus T^{0,2})_{\mathbb{R}}}$ is positive definite). Define $J : V \rightarrow V$, $v \mapsto f_1 f_2 v$, then we see that $J^2 = -\text{id}$. Now we can define a homomorphism of algebraic groups

$$h_s : \text{U}(1) \rightarrow \text{GL}(V)_{\mathbb{R}}, \quad \exp(xi) \mapsto \exp(xJ),$$

and this induces the Kuga–Satake Hodge structure. One can check that h_s is independent of the choice of f_1, f_2 (see [vG3, Lemma 5.5]).

A polarization of the Kuga–Satake Hodge structure. By choosing $e_1, e_2 \in T$ orthogonal with $q(e_i) > 0$ (such e_i exist since the signature of q is $(2+, (\dim(T) - 2)-)$) one defines a polarization of V in the following way:

$$V \times V \rightarrow \mathbb{Q}, \quad (v, w) \mapsto \text{tr}(\pm(e_1 e_2 \iota(v)w)),$$

where $\text{tr}(a)$ for $a \in V$ is the trace of the K -linear map $V \rightarrow V$, $v \mapsto a \cdot v$ and the sign has to be chosen in a suitable way (see [vG3, Prop. 5.9]).

The inclusion $T \subset V \otimes V$. Choose an element $t_0 \in T$ which is invertible in $C(q)$. Then we define a map

$$T(1) \rightarrow \text{End}(V), \quad t \mapsto \{v \mapsto tv t_0\}$$

It is easily seen that this is an injective homomorphism of Hodge structures. Moreover, up to an isomorphism of the Hodge structure V , this map is independent of the choice of t_0 . Composing this inclusion with the isomorphism of Hodge structures induced by the polarization $\text{End}(V) \simeq (V \otimes V)(1)$ yields the inclusion of Hodge structures which we announced.

2.1.5 Corestriction of algebras

Let E/K be a finite, separable extension of fields of degree d and let A be an E -algebra. We use the notations of Section 1.1.5 with $F = E$, so \tilde{E} is a normal closure of E over K , $\sigma_1, \dots, \sigma_d$ is a set of representatives of G/H where $G = \text{Gal}(\tilde{E}/K)$ and $H = \text{Gal}(\tilde{E}/E)$.

For $\sigma \in G$ define the twisted \tilde{E} -algebra as the ring

$$A_{\sigma} := A \otimes_E \tilde{E}$$

which carries an \tilde{E} -algebra structure given by

$$\lambda \cdot (a \otimes e) = a \otimes \sigma^{-1}(\lambda)e.$$

Note that $A_\sigma \simeq A \otimes_E E_\sigma$.

Let V be an E -vector space and W an \tilde{E} -vector space, let $\sigma \in G$. A homomorphism of K -vector spaces $\varphi : V \rightarrow W$ is called σ -linear if $\varphi(\lambda v) = \sigma(\lambda)\varphi(v)$ for all $v \in V$ and $\lambda \in E$. If both, V and W are \tilde{E} -vector spaces, there is a similar notion of an σ -linear homomorphism.

Lemma 2.1.5.1. *The map*

$$\kappa_\sigma : A \rightarrow A_\sigma, \quad a \mapsto a \otimes 1$$

is a σ -linear ring homomorphism and the pair $(A_\sigma, \kappa_\sigma)$ has the following universal property: For all \tilde{E} -algebras B and for all σ -linear ring homomorphisms $\varphi : A \rightarrow B$ there exists a unique \tilde{E} -algebra homomorphism $\tilde{\varphi} : A_\sigma \rightarrow B$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\kappa_\sigma} & A_\sigma \\ & \searrow \varphi & \downarrow \tilde{\varphi} \\ & & B \end{array}$$

commutative.

Proof. We only check the universal property. To give a K -linear map $\alpha : A \otimes_E \tilde{E} \rightarrow B$ is the same as to give a K -bilinear map $\beta : A \times \tilde{E} \rightarrow B$ satisfying

$$\beta(\lambda a, e) = \beta(a, \lambda e) \tag{2.2}$$

for all $a \in A, e \in \tilde{E}$ and $\lambda \in E$. These maps are related by the condition

$$\alpha(a \otimes e) = \beta(a, e).$$

Now given φ as in the lemma, we define

$$\psi : A \times \tilde{E} \rightarrow B, \quad (a, e) \mapsto \sigma(e)a.$$

This is a K -bilinear map which satisfies (2.2) and therefore, it induces a K -linear map

$$\tilde{\varphi} : A \otimes_E \tilde{E} \rightarrow B, \quad a \otimes e \mapsto \sigma(e)\varphi(a).$$

It is clear that $\tilde{\varphi}$ is a ring homomorphism and that it respects the \tilde{E} -algebra structures if we interpret $\tilde{\varphi}$ as a map $\tilde{\varphi} : A_\sigma \rightarrow B$. The uniqueness of this map is immediate. \square

Remark. (i) The lemma shows that up to unique \tilde{E} -algebra isomorphism, the twisted algebra A_{σ_i} depends only on the coset $\sigma_i H$. Indeed, for $\sigma \in \sigma_i H$ the

inclusion $A \hookrightarrow A_{\sigma_i}$ is σ -linear because σ and $\tilde{\sigma}_i$ induce the same embedding of E into \tilde{E} . By the lemma, there exists an \tilde{E} -algebra isomorphism $\alpha_{\sigma, \sigma_i} : A_{\sigma} \xrightarrow{\sim} A_{\sigma_i}$, $a \otimes e \mapsto a \otimes \sigma_i^{-1} \sigma(e)$.

(ii) In Section 1.1.5 we were in the situation $E = \mathbb{Q}(\alpha)$. There we discussed the splitting $E \otimes_{\mathbb{Q}} \tilde{E} \simeq \bigoplus_i \tilde{E}[X]/(X - \sigma_i(\alpha)) \simeq \bigoplus_i E_{\sigma_i}$ and we used the symbol E_{σ_i} for the field \tilde{E} with E -action via $e(x) = \sigma_i(e) \cdot x$. This is precisely our twisted \tilde{E} -algebra E_{σ_i} on which E acts via the inclusion κ_{σ_i} .

For $\tau \in G$ there is a unique τ -linear ring isomorphism $\tau : A_{\sigma_i} \rightarrow A_{\sigma_{\tau i}}$ which extends the identity on $A \subset A_{\sigma_i}$ (in the sense that $\tau \circ \kappa_{\sigma_i} = \kappa_{\sigma_{\tau i}}$). This map is given as the composition of the following two maps: First apply the identity map $A_{\sigma_i} \rightarrow A_{\tau \sigma_i}$, $a \otimes e \mapsto a \otimes e$ which is a τ -linear ring isomorphism. Then apply the isomorphism $\alpha_{\tau \sigma_i, \sigma_{\tau i}}$ (by definition of the G -action on $\{1, \dots, d\}$ we have $\tau \sigma_i \in \sigma_{\tau i} H$). On simple tensors the map τ takes the form

$$a \otimes e \mapsto a \otimes \sigma_{\tau i}^{-1} \tau \sigma_i(e). \quad (2.3)$$

These maps induce a natural action of G on

$$Z_G(A) := A_{\sigma_1} \otimes_{\tilde{E}} \dots \otimes_{\tilde{E}} A_{\sigma_d}$$

where

$$\begin{aligned} & \tau((a_1 \otimes e_1) \otimes \dots \otimes (a_d \otimes e_d)) \\ &= (a_{\tau^{-1}1} \otimes \sigma_1^{-1} \tau \sigma_{\tau^{-1}1}(e_{\tau^{-1}1})) \otimes \dots \otimes (a_{\tau^{-1}d} \otimes \sigma_d^{-1} \tau \sigma_{\tau^{-1}d}(e_{\tau^{-1}d})). \end{aligned} \quad (2.4)$$

By definition, the corestriction of A to K is the K -algebra of G -invariants in $Z_G(A)$

$$\text{Cores}_{E/K}(A) := Z_G(A)^G$$

(see [Dr, §8, Def. 2] or [Ti, 2.2]).

Remark. (i) By [Dr, §8, Cor. 1] there is a natural isomorphism

$$\text{Cores}_{E/K}(A) \otimes_K \tilde{E} \simeq Z_G(A)$$

In particular, with $d = [E : K]$ one gets $\dim_K \text{Cores}_{E/K}(A) = (\dim_E(A))^d$.

(ii) Let $X = \text{Spec}(A)$ for a commutative L -algebra A . Then for any K -algebra B we get a chain of isomorphisms, functorial in B

$$\begin{aligned} \text{Hom}_{K\text{-Alg}}(\text{Cores}_{E/K}(A), B) &\simeq \left(\text{Hom}_{\tilde{E}\text{-Alg}}(Z_G(A), B \otimes_K \tilde{E}) \right)^G \\ &\simeq \text{Hom}_{E\text{-Alg}}(A, B \otimes_K E). \end{aligned}$$

Here, the last isomorphism is given by restricting $f \in \left(\text{Hom}_{\tilde{E}\text{-Alg}}(Z_G(A), B \otimes_K \tilde{E}) \right)^G$ to $A \subset Z_G(A)$, where this inclusion

is given by $a \mapsto a \otimes 1$. (The image of this restriction is contained in the H -invariant part of $B \otimes_K \tilde{E}$ which is $B \otimes_K E$.) This map is an isomorphism, since $Z_G(A)$ is generated as an \tilde{E} -algebra by elements of the form $\sigma(a)$ with $a \in A$ and $\sigma \in G$.

It follows that

$$\mathrm{Res}_{E/K}(\mathrm{Spec}(A)) \simeq \mathrm{Spec}(\mathrm{Cores}_{E/K}(A)),$$

i.e. the Weil restriction of affine E -schemes is the same as the corestriction of commutative E -algebras.

2.1.6 The decomposition theorem

We will assume from now to the end of the section that (T, h, q) is a polarized Hodge structure of primitive K3 type with $E = \mathrm{End}_{\mathrm{Hdg}}(T)$ a totally real number field.

Recall that in this case T is an E -vector space which carries a natural E -valued quadratic form Q (see 1.1.8). It was van Geemen (see [vG4, Prop. 6.3]) who discovered that the algebra $\mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q))$ appears as a sub-Hodge structure in the Kuga–Satake Hodge structure of (T, h, q) . We are going to show that this contains all information on the Kuga–Satake Hodge structure.

Theorem 2.1.6.1. *Denote by (V, h_s) the Kuga–Satake Hodge structure of (T, h, q) .*

(i) *The special Mumford–Tate group of (V, h_s) is the image of $\mathrm{Res}_{E/\mathbb{Q}}(\mathrm{Spin}(Q))$ in $\mathrm{Spin}(q)$ under a morphism m of rational algebraic groups which after base change to \tilde{E} becomes*

$$m_{\tilde{E}} : \begin{cases} \mathrm{Spin}(q_1) \times \dots \times \mathrm{Spin}(q_d) \rightarrow \mathrm{Spin}(q)_{\tilde{E}} \\ (v_1, \dots, v_d) \mapsto v_1 \cdot \dots \cdot v_d. \end{cases}$$

(ii) *Let $W := \mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q))$. Then W can be canonically embedded in V and the image is $\mathrm{SMT}(V)$ -stable and therefore, it is a sub-Hodge structure. Furthermore, there is a (non-canonical) isomorphism of Hodge structures*

$$V \simeq W^{2^{d-1}}.$$

(iii) *We have*

$$\mathrm{End}_{\mathrm{Hdg}}(W) = \mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q))$$

and consequently

$$\mathrm{End}_{\mathrm{Hdg}}(V) = \mathrm{Mat}_{2^{d-1}}(\mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q))).$$

The proof will be given in Section 2.1.8. The theorem tells us that the Kuga–Satake variety A of (T, h, q) is isogenous to a self-product $B^{2^{d-1}}$ of an Abelian variety B with $\mathrm{End}_{\mathbb{Q}}(B) = \mathrm{Cores}_{E/\mathbb{Q}}(C^0(Q))$. Note that B is not simple in general. We will see examples below where B decomposes further.

2.1.7 Galois action on $C(q)_{\tilde{E}}$

By Section 1.1.5 there is a decomposition

$$(T, q)_{\tilde{E}} = \bigoplus_{i=1}^d (T_i, q_i).$$

This in turn yields an isomorphism

$$C(q)_{\tilde{E}} \simeq C(q_{\tilde{E}}) \simeq C(q_1) \widehat{\otimes}_{\tilde{E}} \dots \widehat{\otimes}_{\tilde{E}} C(q_d).$$

If we forget the algebra structure and only look at \tilde{E} -vector spaces, we get

$$C(q)_{\tilde{E}} = \bigoplus_{\mathbf{a} \in \{0,1\}^d} C^{a_1}(q_1) \otimes_{\tilde{E}} \dots \otimes_{\tilde{E}} C^{a_d}(q_d).$$

For $\mathbf{a} \in \{0,1\}^d$ define

$$C^{\mathbf{a}}(q) = C^{a_1}(q_1) \otimes \dots \otimes C^{a_d}(q_d).$$

We introduced an action of $G = \text{Gal}(\tilde{E}/\mathbb{Q})$ on $\{1, \dots, d\}$ (see (1.10)). This induces an action

$$G \times \{0,1\}^d \rightarrow \{0,1\}^d, \quad (\tau, (a_1, \dots, a_d)) \mapsto (a_{\tau^{-1}1}, \dots, a_{\tau^{-1}d}). \quad (2.5)$$

The next lemma describes the Galois action on $C(q)_{\tilde{E}}$.

Lemma 2.1.7.1. (i) *Via the map*

$$C(q_i) \subset C(q)_{\tilde{E}}, \quad v_i \mapsto 1 \otimes \dots \otimes v_i \otimes \dots \otimes 1$$

we interpret $C(q_i)$ as a subalgebra of $C(q)_{\tilde{E}}$. Then the restriction of $\tau \in G$ to $C(q_i)$ induces an isomorphism of $\mathbb{Z}/2\mathbb{Z}$ -graded \mathbb{Q} -algebras

$$\tau : (C(q_i)) \xrightarrow{\sim} C(q_{\tau i}).$$

(ii) *For $\tau \in G$ and $\mathbf{a} \in \{0,1\}^d$ we get*

$$\tau(C^{\mathbf{a}}(q)) = C^{\tau\mathbf{a}}(q).$$

Proof. Tensor the natural inclusion $T \hookrightarrow C(q)$ with \tilde{E} to get a G -equivariant inclusion

$$T \otimes_{\mathbb{Q}} \tilde{E} = \bigoplus_{i=1}^d T_i \rightarrow C(q)_{\tilde{E}}.$$

Using (1.11), we find for $t_i \in C(q_i)$ that $\tau(t_i) \in C(q_{\tau i})$. Now, $C(q_i)$ is spanned as a \mathbb{Q} -algebra by products of the form

$$t_1 \cdot \dots \cdot t_k = \pm(1 \otimes \dots \otimes t_1 \otimes \dots \otimes 1) \cdot \dots \cdot (1 \otimes \dots \otimes t_k \otimes \dots \otimes 1)$$

for $t_1, \dots, t_k \in T_i$. Since G acts by \mathbb{Q} -algebra homomorphisms on $C(q)_{\tilde{E}}$, this implies (i).

Item (ii) is an immediate consequence of (i): The space $C^{\mathbf{a}}(q)$ is spanned as \mathbb{Q} -vector space by products of the form $v_1 \cdots v_d = \pm v_1 \otimes \cdots \otimes v_d$ with $v_i \in C^{a_i}(q_i)$. Then use again, that G acts by \mathbb{Q} -algebra homomorphisms. \square

Lemma 2.1.7.2. *For $i \in \{1, \dots, d\}$ the twisted algebra $C^0(Q)_{\sigma_i}$ is canonically isomorphic as an \tilde{E} -algebra to $C^0(q_i)$. Thus*

$$Z_G(C^0(Q)) \simeq C^0(q_1) \otimes_{\tilde{E}} \cdots \otimes_{\tilde{E}} C^0(q_d).$$

On both sides there are natural G -actions: On the left hand side G acts via the action introduced in (2.4), whereas on the right hand side it acts via the restriction of its action on $C(q)_{\tilde{E}}$ (use Lemma 2.1.7.1). Then the above isomorphism is G -equivariant.

Proof. Fix $i \in \{1, \dots, d\}$. The composition of the canonical inclusion $C^0(Q) \subset C^0(q_1) \simeq C^0(Q)_{\tilde{E}}$ with the restriction to $C^0(Q)$ of the map $\sigma_i : C(q_1) \rightarrow C(q_i)$ from Lemma 2.1.7.1 induces a σ_i -linear ring homomorphism

$$\varphi_i : C^0(Q) \hookrightarrow C^0(q_i).$$

By Lemma 2.1.5.1 we get an \tilde{E} -algebra homomorphism

$$\tilde{\varphi}_i : C^0(Q)_{\sigma_i} \rightarrow C^0(q_i).$$

Recall that there are inclusions $\iota_i : T \hookrightarrow T_i$ (see (1.13)) which satisfy $\sigma \circ \iota_i = \iota_{\sigma_i}$ (see 1.14)). Let $t_1, \dots, t_m \in T$ such that $\iota_1(t_1), \dots, \iota_1(t_m)$ form a q_1 -orthogonal basis of T_1 . Then the vectors $\iota_i(t_1), \dots, \iota_i(t_m)$ form an q_i -orthogonal basis of T_i (use (1.15)). By definition of $\tilde{\varphi}_i$

$$\tilde{\varphi}_i (\iota_1(t_1)^{i_1} \cdots \iota_1(t_m)^{i_d}) = \iota_i(t_1)^{i_1} \cdots \iota_i(t_m)^{i_m}. \quad (2.6)$$

This implies that $\tilde{\varphi}_i$ maps an \tilde{E} -basis of $C(q_1)$ onto an \tilde{E} -basis of $C(q_i)$, whence $\tilde{\varphi}_i$ is an isomorphism of \tilde{E} -algebras.

As for the G -equivariance, we have to check that for all $\tau \in G$ the diagram

$$\begin{array}{ccc} C^0(Q)_{\sigma_i} & \xrightarrow{\tilde{\varphi}_i} & C^0(q_i) \\ \tau \downarrow & & \downarrow \tau \\ C^0(Q)_{\sigma_{\tau i}} & \xrightarrow{\tilde{\varphi}_{\tau i}} & C^0(q_{\tau i}) \end{array}$$

is commutative. It is enough to check this on an \tilde{E} -basis of $C^0(Q)_{\sigma_i}$ because the vertical maps are both τ -linear whereas the horizontal ones are \tilde{E} -linear.

Since $\tau : C^0(Q)_{\sigma_i} \rightarrow C^0(Q)_{\sigma_{\tau i}}$ was defined as the extension of the identity map on $C^0(Q) \subset C^0(Q)_{\sigma_i}$, we have

$$\begin{aligned}
\tilde{\varphi}_{\tau i} \circ \tau (\iota_1(t_1)^{i_1} \cdot \dots \cdot \iota_m(t_m)^{i_m}) &= \tilde{\varphi}_{\tau i} (\iota_1(t_1)^{i_1} \cdot \dots \cdot \iota_m(t_m)^{i_m}) \\
&= \iota_{\tau i}(t_1)^{i_1} \cdot \dots \cdot \iota_{\tau i}(t_m)^{i_m} \\
&= (\tau \circ \iota_i)(t_1)^{i_1} \cdot \dots \cdot (\tau \circ \iota_i)(t_m)^{i_m} \\
&= \tau (\iota_i(t_1)^{i_1} \cdot \dots \cdot \iota_i(t_m)^{i_m}) \\
&= \tau \circ \tilde{\varphi}_i (\iota_1(t_1)^{i_1} \cdot \dots \cdot \iota_m(t_m)^{i_m}).
\end{aligned}$$

This completes the proof of the lemma. \square

2.1.8 Proof of the decomposition theorem

Proof of (i). Recall from 2.1.2 that the spin representation realizes $\text{Spin}(q)$ via left multiplication as a subgroup of $\text{GL}(C^0(q))$. By [vG3, Prop. 6.3], there is a commutative diagram

$$\begin{array}{ccc}
\text{U}(1) & \xrightarrow{h_s} & \text{Spin}(q)_{\mathbb{R}} \\
\parallel & & \downarrow \rho \\
\text{U}(1) & \xrightarrow{h} & \text{SO}(q)_{\mathbb{R}}.
\end{array} \tag{2.7}$$

(Van Geemen works with the Mumford–Tate group, therefore he gets a factor t^2 in 6.3.2. This factor is 1 if one restricts the attention to the special Mumford–Tate group, moreover it is then clear that $h_s(\mathbb{C}^*) \subset \text{CSpin}(q) = \{v \in C^0(q)^* \mid vTv^{-1} \subset T\}$ implies $h_s(\text{U}(1)) \subset \text{Spin}(q)$.)

Claim: There is a Cartesian diagram

$$\begin{array}{ccc}
\text{SMT}(V) & \longrightarrow & \text{Spin}(q) \\
\rho|_{\text{SMT}(V)} \downarrow & & \downarrow \rho \\
\text{SMT}(T) & \longrightarrow & \text{SO}(q).
\end{array}$$

where the horizontal maps are appropriate factorizations of the inclusions $\text{SMT} \subset \text{GL}$ whose existence is guaranteed by (2.7) resp. by Theorem 1.1.8.1.

Proof of the claim. It is clear by looking at (2.7) and the definition of the special Mumford–Tate group that

$$\text{SMT}(V) \subset \text{SMT}(T) \times_{\text{SO}(q)} \text{Spin}(q).$$

In the same way we see that

$$\text{SMT}(T) \subset \rho(\text{SMT}(V))$$

and hence we have a chain of inclusions

$$\mathrm{SMT}(V) \subset \mathrm{SMT}(T) \times_{\mathrm{SO}(q)} \mathrm{Spin}(q) \subset \rho(\mathrm{SMT}(V)) \times_{\mathrm{SO}(q)} \mathrm{Spin}(q).$$

But over any field, the kernel of ρ consists of $\{\pm 1\} \subset \mathrm{SMT}(V)$ (because $h_s(-1) = -1$) and thus

$$\mathrm{SMT}(V) = \rho(\mathrm{SMT}(V)) \times_{\mathrm{SO}(q)} \mathrm{Spin}(q).$$

This proves the claim. (Claim) \square

To continue the proof of (i) we have to define the morphism of rational algebraic groups

$$m : \mathrm{Res}_{E/\mathbb{Q}}(\mathrm{Spin}(Q)) \rightarrow \mathrm{Spin}(q).$$

For that sake, note first that there is a natural isomorphism of \tilde{E} -algebras

$$\begin{aligned} C^0(Q) \otimes_{\mathbb{Q}} \tilde{E} &\simeq C^0(Q) \otimes_E (E \otimes_{\mathbb{Q}} \tilde{E}) \\ &\simeq \bigoplus_i C^0(Q) \otimes_E E_{\sigma_i} \\ &\simeq \bigoplus_i C^0(Q)_{\sigma_i} \\ &\simeq C^0(q_1) \oplus \dots \oplus C^0(q_d) \end{aligned} \tag{2.8}$$

where we use the notations of Section 2.1.5 and for the last identification Lemma 2.1.7.2. Consider the natural G -action on $C^0(q_1) \oplus \dots \oplus C^0(q_d)$ given by

$$(\tau, (v_1, \dots, v_d)) \mapsto (\tau v_{\tau^{-1}1}, \dots, \tau v_{\tau^{-1}d}).$$

On $C^0(Q) \otimes_{\mathbb{Q}} \tilde{E}$, the Galois group G acts by its natural action on \tilde{E} . Then the identification made in (2.8) is G -equivariant and we get an isomorphism of \mathbb{Q} -vector spaces

$$\begin{aligned} C^0(Q) &\simeq (C^0(q_1) \oplus \dots \oplus C^0(q_d))^G \\ v &\mapsto (\sigma_1(v), \dots, \sigma_d(v)). \end{aligned}$$

Now, look at the morphism of \tilde{E} -affine spaces

$$\begin{aligned} C^0(q_1) \oplus \dots \oplus C^0(q_d) &\rightarrow C^0(q)_{\tilde{E}} \\ (v_1, \dots, v_d) &\mapsto v_1 \cdot \dots \cdot v_d. \end{aligned}$$

This morphism is G -equivariant on the \tilde{E} -points and hence it comes from a morphism

$$\mathrm{Res}_{E/\mathbb{Q}} C^0(Q) \rightarrow C^0(q).$$

The restriction of this latter to $\text{Res}_{E/\mathbb{Q}}(\text{Spin}(Q))$ is the morphism m we are looking for. It is a morphism of algebraic groups which after base change to \tilde{E} it takes the form

$$m_{\tilde{E}} : \begin{cases} \text{Res}_{E/\mathbb{Q}}(\text{Spin}(Q))_{\tilde{E}} = \text{Spin}(q_1) \times \dots \times \text{Spin}(q_d) \rightarrow \text{Spin}(q)_{\tilde{E}} \\ (v_1, \dots, v_d) \mapsto v_1 \cdot \dots \cdot v_d. \end{cases}$$

It remains to show that the image of m in $\text{Spin}(q)$ is $\text{SMT}(V)$. Using the claim we have to show that the following diagram exists and that it is Cartesian

$$\begin{array}{ccc} \text{im}(m) & \longrightarrow & \text{Spin}(q) \\ \rho|_{\text{im}(m)} \downarrow & & \downarrow \rho \\ \text{Res}_{E/\mathbb{Q}}(\text{SO}(Q)) & \longrightarrow & \text{SO}(q). \end{array} \quad (2.9)$$

Here, the lower horizontal map is the one coming from Zarhin's Theorem 1.1.8.1.

It is enough to study (2.9) on $\overline{\mathbb{Q}}$ -points. It is easily seen that over $\tilde{E} \subset \overline{\mathbb{Q}}$ the composition $\rho \circ m$ factorizes over

$$\rho_1 \times \dots \times \rho_d : \text{Spin}(q_1) \times \dots \times \text{Spin}(q_d) \rightarrow \text{SO}(q_1) \times \dots \times \text{SO}(q_d) \subset \text{SO}(q)_{\tilde{E}}.$$

This shows that (2.9) exists. Moreover we see that $\rho|_{\text{im}(m)}$ surjects onto $\text{SMT}(T)(\overline{\mathbb{Q}})$ because $\rho_1 \times \dots \times \rho_d$ does so. Since $\ker(\rho) = \{\pm 1\} \subset \text{im}(m)$, the diagram (2.9) is Cartesian. This completes the proof of (i). (i)□

Proof of (ii). Choose $\mathbf{a}_0 = (0, \dots, 0), \dots, \mathbf{a}_r \in \{0, 1\}^d$ such that

$$\left\{ \mathbf{a} \in \{0, 1\}^d \mid \sum_i a_i \equiv 0 \pmod{2} \right\} = G\mathbf{a}_0 \sqcup \dots \sqcup G\mathbf{a}_r,$$

where G acts on $\{0, 1\}^d$ via the action introduced in (2.5). Let $G_{\mathbf{a}_j} \subset G$ be the stabilizer of \mathbf{a}_j . Then

$$\begin{aligned} C^0(q)_{\tilde{E}} &= \bigoplus_{j=0}^r \left(\bigoplus_{[\tau] \in G/G_{\mathbf{a}_j}} C^{\tau\mathbf{a}_j}(q) \right) \\ &= \bigoplus_{j=0}^r D^{\mathbf{a}_j} \end{aligned} \quad (2.10)$$

with $D^{\mathbf{a}_j} = \bigoplus_{[\tau] \in G/G_{\mathbf{a}_j}} C^{\tau\mathbf{a}_j}(q)$.

By Lemma 2.1.7.1 this is a decomposition of G -modules. Moreover, recall that $\text{Spin}(q_1) \times \dots \times \text{Spin}(q_d)$ acts on $C^0(q)_{\tilde{E}}$ by sending (v_1, \dots, v_d) to the endomorphism of $C^0(q)_{\tilde{E}}$ given by left multiplication with $m(v_1, \dots, v_d) =$

$v_1 \cdots v_d$. Under this action each $C^{\mathbf{a}}(q)$ is $(\text{Spin}(q_1) \times \cdots \times \text{Spin}(q_d))$ -stable. Thus, by (i) the decomposition (2.10) is also a decomposition of $\text{SMT}(V)(\tilde{E})$ -modules.

Denote by

$$R := D^{\mathbf{a}_0} = C^{\mathbf{a}_0}(q) = C^0(q_1) \otimes_{\tilde{E}} \cdots \otimes_{\tilde{E}} C^0(q_d).$$

By Lemma 2.1.7.2, using the notations of Section 2.1.5, we have

$$R = Z_G(C^0(Q))$$

as G -modules and hence $R^G = \text{Cores}_{E/\mathbb{Q}}(C^0(Q))$.

Denote by $d_j = \sharp(G/G_{\mathbf{a}_j})$ and choose a set of representatives μ_1, \dots, μ_{d_j} of $G/G_{\mathbf{a}_j}$ in G . We consider three group actions on $R^{\oplus d_j}$:

- First there is a natural $(\text{Spin}(q_1) \times \cdots \times \text{Spin}(q_d))$ -action which is just the diagonal action of the one on R .
- Let $\alpha : G \times R^{\oplus d_j} \rightarrow R^{\oplus d_j}$ be the diagonal action of the G -action on R .
- Finally define the G -action β by

$$\beta : \left\{ \begin{array}{l} G \times \bigoplus_{l=1}^{d_j} R_{[\mu_l]} \rightarrow \bigoplus_{l=1}^{d_j} R_{[\mu_l]} \\ (\tau, (r_{[\mu_1]}, \dots, r_{[\mu_{d_j}]})) \mapsto (\tau r_{[\tau^{-1}\mu_1]}, \dots, \tau r_{[\tau^{-1}\mu_{d_j}]}) \end{array} \right.$$

Now we will proceed in two steps:

(a) We show that $D^{\mathbf{a}_j}$ is isomorphic as G -module and as $(\text{Spin}(q_1) \times \cdots \times \text{Spin}(q_d))$ -module to $R^{\oplus d_j}$ where G acts on the latter via β .

(b) We show that $R^{\oplus d_j}$ is isomorphic as G -module and as $(\text{Spin}(q_1) \times \cdots \times \text{Spin}(q_d))$ -module with G acting via α to $R^{\oplus d_j}$ with G acting via β .

Note that neither of these two isomorphisms is canonical. Once (a) and (b) are proved, we have an isomorphism

$$V_{\tilde{E}} = C^0(q)_{\tilde{E}} \simeq R^{\oplus 2^{d-1}}$$

of G -modules and of $\text{SMT}(V)(\tilde{E})$ -modules, G acting diagonally on the right hand side. Here we use that

$$\sum_j d_j = \sharp \left\{ \mathbf{a} \in \{0, 1\}^d \mid \sum_i a_i \equiv 0 \pmod{2} \right\} = 2^{d-1}.$$

The proof of (ii) is then accomplished by passing to G -invariants.

Proof of (a). Denote by F_j the field $\tilde{E}^{G_{\mathbf{a}_j}}$. As $C^{\mathbf{a}_j}(q) \subset D^{\mathbf{a}_j}$ is $G_{\mathbf{a}_j}$ -stable, $C^{\mathbf{a}_j}(q) = W_j \otimes_{F_j} \tilde{E}$ for some F_j -vector space W_j . Since $C^{\mathbf{a}_j}$ contains units in $C(q)_{\tilde{E}}$, so does $W_j \subset C^{\mathbf{a}_j}$. (Very formally: There is a linear map $C^{\mathbf{a}_j} \rightarrow \text{End}(C(q)_{\tilde{E}})$, $w \mapsto \{v \mapsto v \cdot w\}$ which is defined over F_j . The image of this map over \tilde{E} intersects the Zariski-open subset of automorphisms of $C(q)_{\tilde{E}}$, hence this must happen already over F_j .)

Choose a unit $w_j \in W_j$. Then for $\tau \in G$, since w_j is $G_{\mathbf{a}_j}$ -invariant, $\tau w_j \in C^{\tau \mathbf{a}_j}(q)$ depends only on the coset $\tau G_{\mathbf{a}_j}$ and is again a unit in $C(q)_{\tilde{E}}$.

Define an isomorphism of \tilde{E} -vector spaces

$$\varphi : \begin{cases} D^{\mathbf{a}_j} = \bigoplus_{l=1}^{d_j} C^{\mu_l \mathbf{a}_j}(q) \rightarrow \bigoplus_{l=1}^{d_j} R_{[\mu_l]} \\ (v_{\mu_1}, \dots, v_{\mu_{d_j}}) \mapsto (v_{\mu_1} \cdot \mu_1(w_j), \dots, v_{\mu_{d_j}} \cdot \mu_{d_j}(w_j)). \end{cases}$$

This map is clearly $(\text{Spin}(q_1) \times \dots \times \text{Spin}(q_d))$ -equivariant since this group acts by multiplication on the left whereas we multiply on the right.

As for the G -equivariance (G acting via β on the right hand side), we find for $(v_{[\mu_1]}, \dots, v_{[\mu_{d_j}]}) \in D^{\mathbf{a}_j}$ and $\tau \in G$:

$$\begin{aligned} \varphi(\tau(v_{[\mu_1]}, \dots, v_{[\mu_{d_j}]})) &= \varphi(\tau v_{[\tau^{-1} \mu_1]}, \dots, \tau v_{[\tau^{-1} \mu_{d_j}]}) \\ &= (\tau v_{[\tau^{-1} \mu_1]} \cdot \mu_1 w_j, \dots, \tau v_{[\tau^{-1} \mu_{d_j}]} \cdot \mu_{d_j} w_j) \\ &= (\tau(v_{[\tau^{-1} \mu_1]} \cdot \tau^{-1} \mu_1 w_j), \dots, \tau(v_{[\tau^{-1} \mu_{d_j}]} \cdot \tau^{-1} \mu_{d_j} w_j)) \\ &= \beta(\tau, (v_{\mu_1} \cdot \mu_1 w_j, \dots, v_{\mu_{d_j}} \cdot \mu_{d_j} w_j)) \\ &= \beta(\tau, \varphi(v_{[\mu_1]}, \dots, v_{[\mu_{d_j}]})) \end{aligned}$$

Here we used in the penultimate equality that σw_j depends only on the coset $\sigma G_{\mathbf{a}_j}$. This proves (a). (a)□

Proof of (b). Choose a \mathbb{Q} -basis f_1, \dots, f_{d_j} of F_j . For $i = 1, \dots, d_j$ define an \tilde{E} -vector space homomorphism by

$$\psi_i : \begin{cases} R \hookrightarrow \bigoplus_{l=1}^{d_j} R_{[\mu_l]} \\ r \mapsto (\mu_1(f_i) \cdot r, \dots, \mu_{d_j}(f_i) \cdot r). \end{cases}$$

As $(\text{Spin}(q_1) \times \dots \times \text{Spin}(q_d))(\tilde{E})$ acts by \tilde{E} -linear automorphisms on R , the ψ_i are equivariant for the Spin-action.

Let's show that ψ_i is G -equivariant, G acting on the right hand side via β . For $\tau \in G$ and $r \in R$ we get

$$\begin{aligned}\psi_i(\tau r) &= (\mu_1(f_i) \cdot \tau r, \dots, \mu_{d_j}(f_i) \cdot \tau r) \\ &= (\tau(\tau^{-1}\mu_1(f_i) \cdot r), \dots, \tau(\tau^{-1}\mu_{d_j}(f_i) \cdot r)) \\ &= \beta(\tau, (\mu_1(f_i) \cdot r, \dots, \mu_{d_j}(f_i) \cdot r)) \\ &= \beta(\tau, \psi_i(r)).\end{aligned}$$

Once more, we used the fact that σf_i depends only on the coset $\sigma G_{\mathbf{a}_j}$.

Finally, using Artin's independence of characters (see [L, Thm. VI.4.1]), we get

$$\det((\mu_l(f_i))_{l,i}) \neq 0.$$

Consequently, the map

$$\bigoplus_{i=1}^{d_j} \psi_i : R^{\oplus d_j} \rightarrow R^{\oplus d_j}$$

is an isomorphism which has the equivariance properties we want and (b) is proved. (b) and (ii) \square

Proof of (iii). By (1.4) we have to show that

$$\text{End}_{\text{SMT}(V)}(W) = \text{Cores}_{E/\mathbb{Q}}(C^0(Q)).$$

Denote by \mathfrak{g} the Lie algebra of $\text{SMT}(V)$. Then

$$\begin{aligned}\text{End}_{\text{SMT}(V)}(W) &= \text{End}_{\mathfrak{g}}(W) \\ &= \{f \in \text{End}_{\mathbb{Q}}(W) \mid Xf - fX = 0 \text{ for all } X \in \mathfrak{g}\}.\end{aligned}$$

Since for any field extension K/\mathbb{Q} we have $\text{Lie}(\text{SMT}(V)_K) = \mathfrak{g} \otimes_{\mathbb{Q}} K$ this implies that

$$\text{End}_{\text{SMT}(V)_K}(W_K) = \text{End}_{\text{SMT}(V)}(W) \otimes_{\mathbb{Q}} K. \quad (2.11)$$

Now $\text{SMT}(V)(\tilde{E}) = \text{Spin}(q_1) \times \dots \times \text{Spin}(q_d)(\tilde{E})$ acts on $W_{\tilde{E}} = C^0(q_1) \otimes \dots \otimes C^0(q_d)$ by factorwise left multiplication:

$$((v_1, \dots, v_d), w_1 \otimes \dots \otimes w_d) \mapsto (v_1 \cdot w_1) \otimes \dots \otimes (v_d \cdot w_d).$$

Therefore, using multiplication on the right, we get an inclusion

$$\begin{aligned}(C^0(q_1) \otimes \dots \otimes C^0(q_d))^{\text{op}} &\hookrightarrow \text{End}_{\text{SMT}(V)(\tilde{E})}(W_{\tilde{E}}) \\ w &\mapsto \{w' \mapsto w' \cdot w\}.\end{aligned}$$

Now, $(C^0(q_1) \otimes \dots \otimes C^0(q_d))^{\text{op}} \simeq C^0(q_1)^{\text{op}} \otimes \dots \otimes C^0(q_d)^{\text{op}} \simeq C^0(q_1) \otimes \dots \otimes C^0(q_d)$ and hence passing to G -invariants we have an inclusion

$$\text{Cores}_{E/\mathbb{Q}}(C^0(Q)) \hookrightarrow \text{End}_{\text{SMT}(V)(\mathbb{Q})}(W). \quad (2.12)$$

We will now show that this is an isomorphism over \tilde{E} . Using (2.11) and comparing dimensions this will prove (iii).

To show that (2.12) is an isomorphism over \tilde{E} we have to determine the $\text{Spin}(q_1) \times \dots \times \text{Spin}(q_d)$ -invariants in

$$\text{End}_{\tilde{E}}(C^0(q_1) \otimes \dots \otimes C^0(q_d)) = \text{End}_{\tilde{E}} C^0(q_1) \otimes \dots \otimes \text{End}_{\tilde{E}} C^0(q_d).$$

Using the next lemma inductively, this is equal to

$$\text{End}_{\text{Spin}(q_1)} C^0(q_1) \otimes \dots \otimes \text{End}_{\text{Spin}(q_d)} C^0(q_d).$$

Now by [vG3, Lemma 6.5], $\text{End}_{\text{Spin}(q_i)} C^0(q_i) = C^0(q_i)$. This proves (iii). \square

Lemma 2.1.8.1. *Let G and H be two reductive linear algebraic groups over a field K of characteristic 0. Let M resp. N be finite-dimensional representations over K of G resp. H . Then*

$$(M \otimes_K N)^{G \times H} = M^G \otimes_K N^H.$$

Proof. Decompose $M = \bigoplus_i M_i$ and $N = \bigoplus_j N_j$ in irreducible representations. Then $M_i \otimes N_j$ is an irreducible representation of $G \times H$ since fixing $0 \neq m_0 \in M_i$ and $0 \neq n_0 \in N_j$ the orbit $(G \times H)m_0 \otimes n_0$ generates $M_i \otimes N_j$.

To conclude the proof note that the space of invariants is the direct sum of trivial one-dimensional, irreducible sub representations. \square

2.1.9 Central simple algebras

In this section we review quickly the definition of the Brauer group of a field and state the famous Brauer–Hasse–Noether theorem. Good references for the basic definitions here are [GS, Ch. 2] and [Dr].

Let k be a field of characteristic $\neq 2$. A *central simple algebra* over k is a finite-dimensional k -algebra with center k that has no nontrivial two-sided ideals. By Wedderburn’s theorem, any central simple k -algebra is of the form $M_n(D)$ where n is an integer and D a uniquely determined central division algebra over k .

Let $A \simeq M_n(D)$ and $B \simeq M_l(D')$ be two central simple algebras where D and D' are central division algebras. The algebras A and B are called *Brauer-equivalent* (write $A \sim B$) if $D \simeq D'$. In this way we obtain an equivalence relation on the set of central division algebras over k . The set of equivalence classes carries a natural structure of a commutative group, induced by the tensor product of k -algebras. The unit element is the class of $\text{Mat}_r(k)$ for any $r \geq 1$. The resulting group is called the *Brauer group of k* .

The class of any central simple algebra A in the Brauer group has finite order, this number is called the *exponent of A* and denoted by $e(A)$. Write

$A \simeq \text{Mat}_n(D)$ for a central division algebra D , let d^2 be the k -dimension of D (the dimension of any central simple algebra is a square since it becomes isomorphic to a matrix algebra after base change). Then d is the *index* of A , denoted by $i(A)$. We have $e(A) \mid i(A)$.

Let K/k be a cyclic extension of degree n , let σ be a generator of the Galois group $\text{Gal}(K/k)$, let $a \in k^*$. There is a central simple k -algebra $(\sigma, a, K/k)$ which as a k -algebra is generated by K and an element $y \in (\sigma, a, K/k)$ such that

$$y^n = a \text{ and } r \cdot y = y \cdot \sigma(r) \text{ for } r \in K.$$

This algebra is called the *cyclic algebra associated with σ, a and K/k* . A cyclic algebra over k of dimension 4 is a quaternion algebra.

Theorem 2.1.9.1 (Brauer, Hasse, Noether [BHN]). *Let k be an algebraic number field. Then any central division algebra A over k is a cyclic algebra (for an appropriate cyclic extension K/k and σ and a as above). Moreover, the exponent and the index of A coincide. In particular, a central division algebra of exponent 2 is a quaternion algebra.*

2.1.10 An example

We continue to assume that (T, h, q) is a polarized Hodge structure of primitive K3 type with $E = \text{End}_{\text{Hdg}}(T)$ a totally real number field of degree d over \mathbb{Q} . By Corollary 1.1.8.2 we have $\dim_E T \geq 3$. We will consider now the case that $\dim_E T = 3$.

Then T_1 is a 3-dimensional \tilde{E} -vector space with quadratic form q_1 of signature $(2+, 1-)$. The 3-dimensional quadratic spaces $(T_2, q_2), \dots, (T_d, q_d)$ are negative definite. This implies that

$$\begin{aligned} C^0(q_1)_{\mathbb{R}} &= \text{Mat}_2(\mathbb{R}) \text{ and} \\ C^0(q_i)_{\mathbb{R}} &= \mathbb{H} \text{ for } i \geq 2 \end{aligned}$$

(see [vG3, Thm. 7.7]). Since

$$\text{Cores}_{E/\mathbb{Q}}(C^0(Q)) \otimes_{\mathbb{Q}} \tilde{E} = Z_G(C^0(Q)) = C^0(q_1) \otimes_{\tilde{E}} \dots \otimes_{\tilde{E}} C^0(q_d)$$

we get

$$\text{Cores}_{E/\mathbb{Q}}(C^0(Q)) \otimes_{\mathbb{Q}} \mathbb{R} = \text{Mat}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H} \otimes_{\mathbb{R}} \dots \otimes_{\mathbb{R}} \mathbb{H}.$$

Now, since $\mathbb{H} \otimes \mathbb{H} \simeq \text{Mat}_4(\mathbb{R})$ this becomes

$$\text{Cores}_{E/\mathbb{Q}}(C^0(Q)) \otimes_{\mathbb{Q}} \mathbb{R} \simeq \begin{cases} \text{Mat}_{2^{d-1}}(\mathbb{H}) & \text{for even } d \\ \text{Mat}_{2^d}(\mathbb{R}) & \text{for odd } d. \end{cases} \quad (2.13)$$

On the other hand, the corestriction induces a homomorphism of Brauer groups

$$\text{cores} : \text{Br}(E) \rightarrow \text{Br}(\mathbb{Q})$$

(cf. [Dr, §9, Thm. 5]). By the Brauer–Hasse–Noether Theorem 2.1.9.1 there exists a (possibly split) quaternion algebra D over \mathbb{Q} with

$$\text{Cores}_{E/\mathbb{Q}}(C^0(Q)) \simeq \text{Mat}_{2^{d-1}}(D). \quad (2.14)$$

Combining (2.13) with (2.14) we see that D is a definite quaternion algebra over \mathbb{Q} in case d is even and an indefinite quaternion algebra in case d is odd. The endomorphism algebra of a Kuga–Satake variety of (T, h, q) is $\text{Mat}_{2^{2d-2}}(D)$. Since the dimension of a Kuga–Satake variety is $2^{\dim_{\mathbb{Q}}(T)-2} = 2^{3d-2}$, we have proved

Corollary 2.1.10.1. *Let (T, q, h) be a polarized Hodge structure of primitive K3 type with $E = \text{End}_{\text{Hdg}}(T)$ a totally real number field of degree d over \mathbb{Q} . Assume that $\dim_E(T) = 3$. Then for any Kuga–Satake variety A of (T, h, q) there exists an isogeny*

$$A \sim B^{2^{2d-2}}$$

where B is a 2^d -dimensional Abelian variety.

If d is even, B is a simple Abelian variety of type III, i.e. $\text{End}_{\mathbb{Q}}(B) = D$ for a definite quaternion algebra D over \mathbb{Q} .

If d is odd, B has endomorphism algebra $\text{End}_{\mathbb{Q}}(B) = D$ for an indefinite (possibly split) quaternion algebra D over \mathbb{Q} .

Remark. (i) In the case $d = 2$ and $\dim_E(T) = 3$, van Geemen showed in [vG4, Prop. 5.7] that the Kuga–Satake variety of T is isogenous to a self-product of an Abelian fourfold with definite quaternion multiplication and Picard number 1. It is this case which will be of interest in the next section.

(ii) The case $d = \dim_E(T) = 3$ was also treated by van Geemen (see [vG4, 5.8 and 6.4]). He considers the case $D \simeq \text{Mat}_2(\mathbb{Q})$ and relates this to work of Mumford and Galluzzi. Note that in this case the Abelian variety B of the corollary is not simple.

Example. In [vG4, 3.4], van Geemen constructs concretely a one-dimensional family of six-dimensional K3 type Hodge structures with real multiplication by a quadratic field $E = \mathbb{Q}(\sqrt{d})$ for some square-free integer $d > 0$ which can be written in the form $d = c^2 + e^2$ for rational $c, e > 0$. These Hodge structures are realized as the transcendental lattice of certain K3 surfaces which are double covers of \mathbb{P}^2 , see section 2.2. Pick a member S of this family. Then $T(S) \otimes_{\mathbb{Q}} E$ splits in the direct sum of two three-dimensional E -vector spaces T_1 and T_2 . It turns out that the quadratic space $(T_1, q_1) = (T_1, Q)$ is isometric to $(E^3, \sqrt{d}X_1^2 + \sqrt{d}X_2^2 - (d - \sqrt{d}c)X_3^2)$. Consequently

$$C^0(Q) = (-d, \sqrt{d}(d - \sqrt{d}c))_E \simeq (-1, \sqrt{d} - c)_E.$$

Here for $a, b \in E^*$, the symbol $(a, b)_E$ denotes the quaternion algebra over E generated by elements $1, i$ and j subject to the relations $i^2 = a, j^2 = b$ and $ij = -ji$ (see [vG3, Ex. 7.5]).

The projection formula for central simple algebras (see [Ti, Thm. 3.2]) implies that

$$\begin{aligned} \text{Cores}_{E/\mathbb{Q}}(C^0(\mathbb{Q})) &\simeq (-1, N_{E/\mathbb{Q}}(\sqrt{d} - c))_{\mathbb{Q}} \\ &\simeq (-1, c^2 - d)_{\mathbb{Q}} \simeq (-1, -e^2)_{\mathbb{Q}} \simeq (-1, -1)_{\mathbb{Q}} \end{aligned}$$

which are simply Hamilton's quaternions over \mathbb{Q} . Hence, a Kuga–Satake A variety for $T(S)$ is isogenous to a self-product B^4 where B is a simple Abelian fourfold with $\text{End}_{\mathbb{Q}}(B) = (-1, -1)_{\mathbb{Q}}$.

2.2 Double covers of \mathbb{P}^2 branched along six lines

In this section we use the results of the preceding section to investigate double covers of \mathbb{P}^2 which are branched along six lines. These K3 surfaces have been studied extensively by various people. Paranjape [P] discovered a very beautiful geometric explanation of the Kuga–Satake correspondence. Matsumoto, Sasaki and Yoshida [MSY] studied in a long paper the period map and its monodromy for the four-dimensional family of such surfaces.

We start by recalling the description of the transcendental lattice and the image of the period map. We use this to list all possible endomorphism algebras of the transcendental lattice of such a surface. As already van Geemen showed, the moduli space of such double covers contains countably many curves parameterizing K3 surfaces with real multiplication by a quadratic totally real field. As we know by a result of Lombardo that the Kuga–Satake variety of a K3 surface in our family is of Weil type, we include a subsection on such Abelian varieties. The decomposition theorem of the preceding section then implies that in the case of real multiplication the Kuga–Satake variety has quaternion multiplication. Luckily, the space of quaternion Weil cycles for these Abelian varieties is known to be algebraic in view of work of Schoen and van Geemen. We apply this to prove the main result of this chapter which says that the Hodge conjecture is true for self-products of K3 surfaces which are double covers of \mathbb{P}^2 ramified along six lines.

2.2.1 The transcendental lattice

Consider a K3 surface S with a finite morphism $p : S \rightarrow \mathbb{P}^2$ such that the branch locus of π is the union of six lines in general position.

The Néron–Severi group of S contains the 15 classes e_1, \dots, e_{15} corresponding to the exceptional divisors over the intersection points of the six lines. Let h be the class of the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$.

Define $\tilde{T}(S)_{\mathbb{Z}} := \langle e_1, \dots, e_{15}, h \rangle^{\perp} \subset H^2(S, \mathbb{Z})$. The (integral) transcendental lattice of S is defined to be $T(S)_{\mathbb{Z}} := \text{NS}(S)_{\mathbb{Z}}^{\perp}$. Then we have

$$T(S)_{\mathbb{Z}} \subset \tilde{T}(S)_{\mathbb{Z}}.$$

After tensoring with \mathbb{Q} both, $T(S) = T(S)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\tilde{T}(S)_{\mathbb{Z}} = T(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ are polarized Hodge structures of primitive K3 type. In addition, $T(S)$ is irreducible. We are going to study the possible endomorphism algebras of this below (see 2.2.3).

Denote by U the hyperbolic lattice

$$U := (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).$$

For a lattice L with quadratic form q and for $n \in \mathbb{Z}$, denote by $L(n)$ the free Abelian group L equipped with the quadratic form nq .

Proposition 2.2.1.1 (see [P], Lemma 1 or [MSY], Sect. 0.3). *The lattice \widetilde{T} is isomorphic to $U(2) \oplus U(2) \oplus \langle -2 \rangle \oplus \langle -2 \rangle$.*

Paranjape's proof uses the fact that if the six lines are tangent to a plane quadric, then S is the Kummer surface associated to the Jacobian of a genus two curve.

Matsumoto, Sasaki and Yoshida construct very explicitly differentiable two-cycles on S which represent generators of \widetilde{T} . This is done by identifying S with an elliptic fibration over \mathbb{P}^1 . The morphism $S \rightarrow \mathbb{P}^1$ is defined as the composition of the map $S \rightarrow \mathbb{P}^2$ with the projection of \mathbb{P}^2 onto one of the branch curves.

2.2.2 Moduli

Denote by B the four-dimensional configuration space of six lines in \mathbb{P}^2 (see [MSY, Sect. 0.2] for the precise definition of B). Over B there is a family $\pi : \mathcal{X} \rightarrow B$ of K3 surfaces, where \mathcal{X}_b is the minimal resolution of the double cover of \mathbb{P}^2 ramified along the six lines in \mathbb{P}^2 corresponding to the point $b \in B$. In [MSY], the authors study the monodromy of the map π . By doing so, they are able to determine the image of the period map associated with π .

We define the lattice

$$(\Gamma, q) := U(2) \oplus U(2) \oplus \langle -2 \rangle \oplus \langle -2 \rangle,$$

the set of (-2) -classes will be denoted by

$$\Delta := \{\alpha \in \Gamma \mid q(\alpha) = -2\}.$$

Let $\Omega := \{[\sigma] \in \mathbb{P}(\Gamma_{\mathbb{C}}) \mid q(\sigma) = 0, q(\sigma + \bar{\sigma}) > 0\}$. This is the period domain which parametrizes polarized Hodge structures of primitive K3 type on $(\Gamma, q) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 2.2.2.1 (Matsumoto, Sasaki, Yoshida, see [MSY] Prop. 2.10.1). *The image of the (multivalued) period map*

$$\mathcal{P} : B \rightarrow \Omega$$

is

$$\Omega^0 := \Omega \setminus (\cup_{\alpha \in \Delta} S_{\alpha})$$

where $S_{\alpha} := \{[\sigma] \in \Omega \mid q(\sigma, \alpha) = 0\}$.

Remark. The easy direction of the theorem says that no Hodge structure on Γ is realized by the family π where a class $\alpha \in \Delta$ is of type $(1, 1)$. Indeed, assume that such a Hodge structure comes from a K3 surface lying in the family π . Then this surface carries a holomorphic line bundle with first

Chern class α . Now, by the Riemann–Roch theorem if the first Chern class of a line bundle has square -2 , then either the line bundle or its dual admits a non-trivial global section. But $\alpha \in \Delta$ is orthogonal to $kh + \sum_i e_i$ which is an ample class for $k \gg 0$. This yields a contradiction.

2.2.3 Endomorphisms of the transcendental lattice

For any K3 surface S which is a member of the family π , denote by

$$E_S := \text{End}_{\text{Hdg}}(T(S)).$$

Since $T(S)$ is an E_S vector space and since the dimension m of $T(S)$ over E_S is ≥ 3 in the case that E_S is totally real (see Corollary 1.1.8.2), we have the following possibilities:

(a) $E_S = \mathbb{Q}$, which is the general case.

(b) $E_S = \mathbb{Q}(\sqrt{d})$ for $d \in \mathbb{Z}$ positive and square-free. This is the only case in which we get K3 surfaces with real multiplication by $E_S \neq \mathbb{Q}$. In this case $T(S) \otimes \mathbb{Q}(\sqrt{d})$ splits in a direct sum of three dimensional eigenspaces for the \sqrt{d} -action. There exists a one-dimensional subfamily $\rho : \mathcal{Y} \rightarrow C$ of π with $S = \mathcal{Y}_0$ for some $0 \in C$ such that the general fiber has real multiplication by $\mathbb{Q}(\sqrt{d})$ where multiplication by \sqrt{d} is given everywhere by the same Hodge class as the one on $S \times S$ (use Ehresmann’s theorem to make sense of this). The family C lies over the intersection of Ω^0 with the projectivization of the \sqrt{d} -eigenspace in $T(S)_{\mathbb{C}}$. Over a dense, countable subset of C the fibers have complex multiplication by a purely imaginary quadratic extension of $\mathbb{Q}(\sqrt{d})$.

(c) $E_S = \mathbb{Q}(\sqrt{-d})$ for $d \in \mathbb{Z}$ positive and square-free and $\dim_{\mathbb{Q}(\sqrt{-d})} T(S) = 1$. These surfaces are exceptional, meaning that they have maximal Picard number. The points in B which parametrize surfaces of this type are isolated in the sense that the Hodge class on $H^2(S, \mathbb{Q}) \otimes H^2(S, \mathbb{Q})$ corresponding to $\sqrt{-d}$ does not remain of Hodge type on any point nearby.

(d) $E_S = \mathbb{Q}(\sqrt{-d})$ for $d \in \mathbb{Z}$ positive and square-free and $\dim_{\mathbb{Q}(\sqrt{-d})} T(S) = 2$. These surfaces come in one-dimensional families.

(e) $E_S = \mathbb{Q}(\sqrt{-d})$ for $d \in \mathbb{Z}$ positive and square-free and $\dim_{\mathbb{Q}(\sqrt{-d})} T(S) = 3$. These surfaces appear in two-dimensional families.

(f) $E_S = K$ for a CM-field K of degree 4 over \mathbb{Q} . Then $\dim_K T(S) = 1$ since $\dim_{\mathbb{Q}} T(S) \leq 6$. The points in B which parametrize surfaces of this type are isolated in the sense of (c).

(g) $E_S = K$ for a CM-field K of degree 6 over \mathbb{Q} . This case behaves in the same way as the one discussed in (f).

Note that all cases appear which is easily proved using Proposition 2.2.2.1 (see [vG4], Example 3.4 for an explicit example of case (b)).

2.2.4 Abelian varieties of Weil type

By a result of Lombardo, the Kuga–Satake variety of the polarized Hodge structure of primitive K3 type $\tilde{T}(S)$ is of Weil type. Here we briefly recall what this means.

Let $K = \mathbb{Q}(\sqrt{-d})$ for some square-free $d \in \mathbb{N}$. A polarized Abelian variety (A, H) of dimension $2n$ is said to be of *Weil type for K* if there is an inclusion $K \subset \text{End}_{\mathbb{Q}}(A)$ mapping $\sqrt{-d}$ to φ such that

- the restriction of $\varphi_{\mathbb{C}}^*$ to $H^{1,0}(A)$ is diagonalizable with eigenvalues $\sqrt{-d}$ and $-\sqrt{-d}$ which appear both with multiplicity n ,
- $\varphi^*H = dH$.

There is a natural K -valued Hermitian form on the K -vector space $H^1(A, \mathbb{Q})$ which is defined by

$$\tilde{H} : \begin{cases} H^1(A, \mathbb{Q}) \times H^1(A, \mathbb{Q}) \rightarrow K \\ (v, w) \mapsto H(\varphi^*v, w) + \sqrt{-d}H(v, w). \end{cases}$$

By definition, the discriminant of a polarized Abelian variety of Weil type (A, H, K) is

$$\text{disc}(A, H, K) = \text{disc}(\tilde{H}) \in \mathbb{Q}^*/N(K^*)$$

where $N : K \rightarrow \mathbb{Q}$ is the norm map.

Polarized Abelian varieties of Weil type come in n^2 -dimensional families (see [vG2, 5.3]).

Weil introduced such varieties as examples of Abelian varieties which carry interesting Hodge classes. He constructs a two-dimensional space, called the space of *Weil cycles*

$$W_K \subset H^{n,n}(A, \mathbb{Q}).$$

For the definition of W_K see [vG2, 5.2]. In general, the algebraicity of the classes in W_K is not known. Nonetheless there are some positive results. Here we mention one which we will use below.

Theorem 2.2.4.1 (Schoen [Sch] and van Geemen [vG1], Thm. 3.7). *Let (A, H) be a polarized Abelian fourfold of Weil type for the field $\mathbb{Q}(i)$. Assume that the discriminant of $(A, H, \mathbb{Q}(i))$ is 1. Then the space of Weil cycles $W_{\mathbb{Q}(i)}$ is spanned by classes of algebraic cycles.*

Van Geemen uses a six-dimensional eigenspace in the complete linear system of the line bundle \mathcal{L} which is the unique totally symmetric line bundle with $c_1(\mathcal{L}) = H$ to get a rational (2:1) map of A onto a quadric $Q \subset \mathbb{P}^5$.

Then the projection on $W_{\mathbb{Q}(i)}$ of the classes of the pullbacks of the two rulings of Q generate the space $W_{\mathbb{Q}(i)}$.

2.2.5 Abelian varieties with quaternion multiplication

Let D be a definite quaternion algebra over \mathbb{Q} . Such a D admits an involution $x \mapsto \bar{x}$ which after tensoring with \mathbb{R} becomes the natural involution on Hamilton's quaternions \mathbb{H} .

A polarized Abelian variety (A, H) of dimension $2n$ is said to have *quaternion multiplication by D* if there is an inclusion $D \subset \text{End}_{\mathbb{Q}}(A)$ such that

- $H^1(A, \mathbb{Q})$ becomes a D -vector space and
- for $x \in D$ we have $x^*H = x\bar{x}H$.

We say that (A, H, D) is an Abelian variety of definite quaternion type. Polarized Abelian varieties of dimension $2n$ with quaternion multiplication by the same quaternion algebra come in $n(n-1)/2$ -dimensional families (cf. [BL, Sect. 9.5]).

Let $K \subset D$ be a quadratic extension field of \mathbb{Q} . Then K is a CM field and (A, H, K) is a polarized Abelian variety of Weil type (see [vGV, Lemma 4.5]). The space of quaternion Weil cycles of (A, H, D)

$$W_D \subset H^{n,n}(A, \mathbb{Q})$$

is defined to be the span of x^*W_K where x runs over D . It can be shown that this is independent of the choice of K (see [vGV, Prop. 4.7]). For the general member of the family of polarized Abelian varieties with quaternion multiplication these are essentially all Hodge classes:

Theorem 2.2.5.1 (Abdulali, see [Ab2], Thm. 4.1). *Let (A, H, D) be a general Abelian variety of quaternion type. Then the space of Hodge classes on any self-product of A is generated by products of divisor classes and quaternion Weil cycles on A .*

In particular, if for one quadratic extension field $K \subset D$ the space of Weil cycles W_K is known to be algebraic, then the Hodge conjecture holds for any self-product of A .

In Abdulali's theorem, a triple (A, H, D) is general if the special Mumford–Tate group of $H^1(A, \mathbb{Q})$ is the maximal one. In the moduli space of triples (A, H, D) the locus of general triples is everything but a countable union of proper, closed subsets.

2.2.6 The Kuga–Satake variety

Denote by A the Kuga–Satake variety associated with $\tilde{T}(S)$.

Theorem 2.2.6.1 (Lombardo, see [Lo], Cor. 6.3 and Thm. 6.4). *There is an isogeny*

$$A \sim B^4$$

where B is an Abelian fourfold with $\mathbb{Q}(i) \subset \text{End}_{\mathbb{Q}}(B)$. Moreover, B admits a polarization H such that $(B, H, \mathbb{Q}(i))$ is a polarized Abelian variety of Weil type with $\text{disc}(B, H, \mathbb{Q}(i)) = 1$.

Paranjape [P] explains in a very nice way how this variety B is geometrically related to S . He shows that there exists a triple

$$(C, E, f : C \rightarrow E)$$

where C is a genus five curve, E an elliptic curve and f a $(4 : 1)$ map such that

$$\text{Prym}(f) = B.$$

Then S can be obtained as the resolution of a certain quotient of $C \times C$. It is noteworthy that Paranjape does not construct explicitly a triple (C, E, f) starting with a K3 surface S in the family π . His proof goes the other way round. He associates to any triple a K3 surface and shows then that letting vary the triple he obtains all surfaces in the family π .

Paranjape's construction establishes that the Kuga–Satake inclusion

$$\tilde{T}(S) \hookrightarrow H^2(B^4 \times B^4, \mathbb{Q}) \quad (2.15)$$

is given by an algebraic cycle on $S \times B^4 \times B^4$.

2.2.7 Proof of Theorem 2

In Section 1.1.2 we have seen that the space of Hodge classes on a self-product of a K3 surface S whose algebraicity is not known by standard arguments can be identified with the space $\text{End}_{\text{Hdg}}(T(S))$.

In the case of the surfaces considered in this section we can use the results on the Kuga–Satake variety to show

Theorem 2. *Let S be a K3 surface which is a double cover of \mathbb{P}^2 ramified along six lines. Then the Hodge conjecture is true for $S \times S$.*

Proof. We have to prove that $E_S := \text{End}_{\text{Hdg}}(T(S))$ is spanned by algebraic classes. Since the Picard number of S is greater than or equal to 16, we can apply Mukai's Theorem 1.1.4.1 which tells us that any isometry in E_S is algebraic. By Corollary 1.1.4.2 we are done if S has complex multiplication.

Therefore, we may assume that S has real multiplication. Looking back to the list in 2.2.3, this means that either $E_S = \mathbb{Q}$ or $E_S = \mathbb{Q}(\sqrt{d})$. In the first case we use the fact, that the class of the diagonal $\Delta \subset S \times S$ induces the identity on the cohomology and that the Künneth projectors are algebraic on surfaces so that $\mathbb{Q} \text{id} \subset E_S$ is spanned by an algebraic class.

It remains to study the case $E_S = \mathbb{Q}(\sqrt{d})$. The idea is to consider the Kuga–Satake variety $A(S)$ of $\tilde{T}(S) = T(S)$. By Paranjape’s theorem the inclusion

$$\tilde{T}(S) \subset H^2(A(S) \times A(S), \mathbb{Q})$$

is algebraic. It follows that there is an algebraic projection $\pi : H^2(A(S) \times A(S), \mathbb{Q}) \rightarrow \tilde{T}(S)$ (see [Kl, Cor. 3.14]) and therefore it is enough to show that there is an algebraic class

$$\alpha \in H^2(A(S) \times A(S), \mathbb{Q}) \otimes H^2(A(S) \times A(S), \mathbb{Q}) \subset H^4(A(S)^4, \mathbb{Q})$$

with $\pi \otimes \pi(\alpha) = \sqrt{d}$.

Combining Corollary 2.1.10.1 with Lombardo’s theorem 2.2.6.1 we see that $A(S) \sim B^4$ where B is an Abelian fourfold with $\text{End}_{\mathbb{Q}}(B) = D$ for a definite quaternion algebra and $\mathbb{Q}(i) \subset \text{End}_{\mathbb{Q}}(B)$. Moreover, there is a polarization H of B such that $(B, H, \mathbb{Q}(i))$ is a polarized Abelian variety of Weil type of discriminant 1. Since by [BL, Prop. 5.5.7], the Picard number of B is 1, (B, H, D) is a polarized Abelian variety of quaternion type.

There is a one-dimensional family $(B, H, D)_t$ of deformations of (B, H, D) and this corresponds to a one-dimensional family S_t of deformations of S which parametrizes K3 surfaces with real multiplication by the same class. By Abdulali’s Theorem 2.2.5.1, for t general the space of Hodge classes on $(B_t)^{16} \sim A(S_t)^4$ is generated by products of divisors and quaternion Weil cycles, that is by products of H and classes in W_D . Denote the span of these products in $H^4(A(S_t)^4, \mathbb{Q})$ by F_t .

Since the class corresponding to $\sqrt{d} \in \tilde{T}(S_t) \otimes \tilde{T}(S_t)$, the projection $\pi : H^2(A(S_t)^2, \mathbb{Q}) \rightarrow \tilde{T}(S_t)$ and the space F_t are locally constant, there exists a locally constant class $\alpha_t \in H^4(A_{S_t}, \mathbb{Q})$ with the properties:

- for all t we have $\pi \otimes \pi(\alpha_t) = \sqrt{d}$,
- for all t we have $\alpha_t \in F_t$.

Now by Schoen’s and van Geemen’s Theorem 2.2.4.1 the space of Weil cycles $W_{\mathbb{Q}(i)}$ is generated by algebraic classes on any B_t . It follows that W_D is generated by algebraic classes and consequently F_t is generated by algebraic classes for any t . In particular, $\alpha_t \in F_t$ is algebraic. This proves the theorem. \square

Remark. We note that this proof does only use the part of Mukai’s result for which a rigorous proof has been written up.

Chapter 3

Hilbert schemes of points on K3 surfaces

This chapter contains some calculations of algebraic classes on compact Hyperkähler fourfolds which are deformation equivalent to the second Hilbert square of a K3 surface. First we show in Section 3.1 that if S is a K3 surface, then the Hodge conjecture for $\text{Hilb}^2(S)$ is equivalent to the Hodge conjecture for $S \times S$. Motivated by this, we try to construct algebraic cycles on $\text{Hilb}^2(S)$ and on deformations of $\text{Hilb}^2(S)$. Having found interesting cycles on such deformations, one could try to deform them to algebraic cycles on Hilbert schemes.

In Section 3.2, we calculate the Chern character of the tautological vector bundle $\mathcal{L}^{[2]}$ on $\text{Hilb}^2(S)$ associated with a line bundle $\mathcal{L} \in \text{Pic}(S)$. We show that if $h^0(\mathcal{L}) \geq 2$, then the bundle $\mathcal{L}^{[2]}$ is μ -stable with respect to an appropriate polarization of $\text{Hilb}^2(S)$. This could be of interest for the deformation of algebraic cycles along twistor lines by making use of Verbitsky's theory of hyperholomorphic bundles (see Section 1.2.3).

In Section 3.3, we consider the Fano variety of lines on a cubic fourfold. A result of Beauville and Donagi says that this variety is a deformation of $\text{Hilb}^2(S)$. More precisely, they show that to a K3 surface S of degree 14 in \mathbb{P}^8 one can associate a cubic fourfold $Y \subset \mathbb{P}^5$. If S and Y are general, then $\text{Hilb}^2(S) \simeq F(Y)$ where $F(Y)$ is the Fano variety of lines on Y . Using the universal line, we have a correspondence between the cubic and the Fano variety which can be used to transform cycles on the cubic to cycles on the Fano variety. We calculate the fundamental class of the surface parametrizing lines which meet a given curve in Y and the fundamental class of the Fano surface of lines contained in a hypersurface of Y .

Finally, in Section 3.4 we discuss the results in view of our question. It turns out that all algebraic classes we constructed before, are linear combinations of products of divisor classes and of the second Chern class of the variety. We give a conceptual explanation of this.

3.1 The cohomology of the Hilbert square

3.1.1 The cohomology ring

Let S be a smooth, projective surface. As a preparation for the calculations below we review the cohomology ring of $X := \text{Hilb}^2(S)$. For simplicity we will assume $H^1(S) = 0$ which is the case of interest for us since we study K3 surfaces.

We now fix some notation which will be used in this and in the next section where we calculate the Chern character of tautological bundles. Let

- $\iota_\Delta : \Delta \hookrightarrow S \times S$ be the diagonal.
- $\sigma : \widetilde{S \times S} \rightarrow S \times S$ be the blowup of $S \times S$ in Δ . The natural action of the symmetric group \mathfrak{S}_2 on $S \times S$ extends to a holomorphic action on $\widetilde{S \times S}$ and $\text{Hilb}^2(S) = \widetilde{S \times S} / \mathfrak{S}_2$.
- $\iota_D : D \hookrightarrow \widetilde{S \times S}$ be the exceptional divisor of σ . Recall that $D \simeq \mathbb{P}(\mathcal{N}_{\Delta|S \times S}) \simeq \mathbb{P}(\mathcal{T}_S)$ is a projective bundle over $\Delta \simeq S$.
- In the following diagram we draw a picture of our situation and, at the same time, give names to the various natural maps.

$$\begin{array}{ccccc}
 D & \xrightarrow{\iota_D} & \widetilde{S \times S} & \xrightarrow{\pi} & \text{Hilb}^2(S) = X \\
 \sigma_D \downarrow & & \downarrow \sigma & & \downarrow \\
 S \simeq \Delta & \xrightarrow{\iota_\Delta} & S \times S & \xrightarrow{\tilde{r}_2} & \text{Sym}^2(S) \\
 & \nearrow \tilde{r}_1 & \downarrow r_1 & \searrow r_2 & \\
 & & S & & S
 \end{array}$$

By [Gr], we know that via pullback we get an isomorphism

$$\pi^* H^*(\text{Hilb}^2(S), \mathbb{Q}) \xrightarrow{\sim} H^*(\widetilde{S \times S}, \mathbb{Q})^{\mathfrak{S}_2}. \quad (3.1)$$

Therefore, in order to calculate the cohomology ring of $\text{Hilb}^2(S)$, it is enough to determine $H^*(\widetilde{S \times S}, \mathbb{Q})$ and the \mathfrak{S}_2 -action on it.

By [Vo2, Thm. 7.31], we have

$$\begin{aligned}
 H^2(\widetilde{S \times S}, \mathbb{Q}) &= \sigma^* H^2(S \times S, \mathbb{Q}) \oplus \mathbb{Q}[D] \quad \text{and} \\
 H^4(\widetilde{S \times S}, \mathbb{Q}) &= \sigma^* H^4(S \times S, \mathbb{Q}) \oplus \iota_{D,*} \sigma_D^* H^2(S, \mathbb{Q})
 \end{aligned} \quad (3.2)$$

as \mathbb{Q} -vector spaces.

The \mathfrak{S}_2 -action on $\sigma^* H^*(S \times S) = \sigma^*(H^*(S) \otimes H^*(S))$ is given by $\tau(\alpha \otimes \beta) = (-1)^{\deg(\alpha) \deg(\beta)} \beta \otimes \alpha$ since $H^1(S, \mathbb{Q}) = 0$. In both lines of equality (3.2), \mathfrak{S}_2 acts trivially on the second summand on the right.

Thus, we get a natural injective homomorphism

$$\varphi : H^2(S, \mathbb{Q}) \rightarrow H^2(X, \mathbb{Q}), \quad \alpha \mapsto \varphi(\alpha)$$

defined by the equality $(\pi^* \circ \varphi)(\alpha) = \tilde{r}_1^* \alpha + \tilde{r}_2^* \alpha$.

Denote by $\delta \in H^2(X, \mathbb{Q})$ the class with $\pi^* \delta = [D]$. (We will see later that δ is not effective, the fundamental class of the exceptional divisor of the Hilbert–Chow morphism is 2δ (see the proof of Lemma 3.2.2.1).)

The following result is well-known, for the convenience of the reader we include a proof.

Proposition 3.1.1.1. *We have*

$$H^2(X, \mathbb{Q}) = \varphi(H^2(S, \mathbb{Q})) \oplus \mathbb{Q}\delta \quad \text{and} \quad H^4(X, \mathbb{Q}) \simeq \text{Sym}^2 H^2(X, \mathbb{Q}).$$

Proof. In degree two the statement follows immediately from the above observations.

In degree four we have

$$H^4(\widetilde{S \times S}, \mathbb{Q})^{\mathfrak{S}_2} = \sigma^* H^4(S \times S, \mathbb{Q})^{\mathfrak{S}_2} \oplus \iota_{D,*} \sigma_D^* H^2(S, \mathbb{Q}). \quad (3.3)$$

We will show that all \mathfrak{S}_2 -invariant classes in $H^4(\widetilde{S \times S}, \mathbb{Q})$ are linear combinations of products of \mathfrak{S}_2 -invariant classes in $H^2(S \times S, \mathbb{Q})$. This proves that the map $\text{Sym}^2 H^2(X, \mathbb{Q}) \rightarrow H^4(X, \mathbb{Q})$ is surjective. Then the injectivity follows by an easy dimension count.

To begin with, note that the left summand in (3.3) is isomorphic to

$$\sigma^* \left(\text{Sym}^2 H^2(S, \mathbb{Q}) \oplus \mathbb{Q}([\Delta]^{0,4} + [\Delta]^{4,0}) \right)$$

where the superscripts to $[\Delta]$ refer to the corresponding Künneth factors. This shows that we only have to represent classes in $\iota_{D,*} \sigma_D^* H^2(S, \mathbb{Q})$ and the class $\sigma^*[\Delta]$ as symmetric products of symmetric classes of degree 2.

For $\alpha \in H^2(S, \mathbb{Q})$

$$\begin{aligned} \iota_{D,*} \sigma_D^* \alpha &= \frac{\iota_{D,*} \{ \iota_D^* (\tilde{r}_1^* \alpha + \tilde{r}_2^* \alpha) \}}{2} \\ &= \frac{(\tilde{r}_1^* \alpha + \tilde{r}_2^* \alpha)}{2} \cup [D] \in \pi^* \text{Sym}^2 H^2(X, \mathbb{Q}). \end{aligned}$$

Here we used that $\tilde{r}_i \circ \iota_D = \sigma_D$ for $i = 1, 2$.

Let $\xi := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{T}_S)}(-1)) \in H^2(D, \mathbb{Q})$, then $[D]|_{[D]} = \xi$. Using the lemma below, we get

$$\begin{aligned} [D]^2 &= \iota_{D,*}([D]|_D) = \iota_{D,*} \xi \\ &= \iota_{D,*} \{ -(-\xi + \sigma_D^* c_1(S)) \} + \iota_{D,*} \sigma_D^* c_1(S) \\ &= \sigma^* \iota_{\Delta,*}(-1_S) + \frac{\iota_{D,*} \{ \iota_D^* (\tilde{r}_1^* c_1(S) + \tilde{r}_2^* c_1(S)) \}}{2} \\ &= -\sigma^*[\Delta] + \frac{\tilde{r}_1^* c_1(S) + \tilde{r}_2^* c_1(S)}{2} \cup [D]. \end{aligned} \quad (3.4)$$

This shows that $\sigma^*[\Delta] \in \pi^* \text{Sym}^2 H^2(X, \mathbb{Q})$ and therefore completes the proof. \square

Lemma 3.1.1.2. *Let $\alpha \in H^k(S, \mathbb{Q})$. Then*

$$\iota_{D,*} \{ \sigma_D^* \alpha \cup (-\xi + \sigma_D^* c_1(S)) \} = \sigma^* \iota_{\Delta,*}(\alpha).$$

Proof. Let $\beta \in H^{8-k-2}(\widetilde{S \times S}, \mathbb{Q})$, write $\beta = \sigma^* \beta' + \iota_{D,*} \gamma$ (this is possible, see (3.2)). By Poincaré duality it is enough to show that

$$\int_{\widetilde{S \times S}} \iota_{D,*} \{ \sigma_D^* \alpha \cup (-\xi + \sigma_D^* c_1(S)) \} \cup \sigma^* \beta' = \int_{\widetilde{S \times S}} \sigma^* \iota_{\Delta,*} \alpha \cup \sigma^* \beta' \quad (3.5)$$

and

$$\int_{\widetilde{S \times S}} \iota_{D,*} \{ \sigma_D^* \alpha \cup (-\xi + \sigma_D^* c_1(S)) \} \cup \iota_{D,*} \gamma = \int_{\widetilde{S \times S}} \sigma^* \iota_{\Delta,*} \alpha \cup \iota_{D,*} \gamma. \quad (3.6)$$

We have

$$\begin{aligned} & \int_{\widetilde{S \times S}} \iota_{D,*} \{ \sigma_D^* \alpha \cup (-\xi + \sigma_D^* c_1(S)) \} \cup \sigma^* \beta' \\ &= \int_D \sigma_D^* \alpha \cup (-\xi + \sigma_D^* c_1(S)) \cup \iota_D^* \sigma^* \beta' \\ &= \int_D \sigma_D^* \alpha \cup (-\xi) \cup \sigma_D^* \iota_{\Delta}^* \beta' + \int_D \sigma_D^* (\alpha \cup c_1(S)) \cup \iota_{\Delta}^* \beta' \\ &= \int_{\Delta} \alpha \cup \beta'_{|\Delta} + 0. \end{aligned}$$

The second summand vanishes for degree reasons and for the first equality we use that for $x \in S \simeq \Delta$

$$\int_{\Delta} [x] = \int_D \sigma_D^* [x] \cup (-\xi).$$

But then

$$\begin{aligned} \int_{\Delta} \alpha \cup \beta'_{|\Delta} &= \int_{S \times S} \iota_{\Delta,*} \alpha \cup \beta' \\ &= \int_{\widetilde{S \times S}} \sigma^* \iota_{\Delta,*} \alpha \cup \sigma^* \beta' \end{aligned}$$

and this proves (3.5).

To prove (3.6) recall that in $H^*(D, \mathbb{Q})$ we have the relation $\xi^2 - \sigma_D^* c_1(S) \xi + \sigma_D^* c_2(S) = 0$ (see [GH, Prop. on p. 606]). Moreover, note that $c_2(S) = \iota_{\Delta}^* [\Delta]$. Using this we obtain

$$\begin{aligned} & \int_{\widetilde{S \times S}} \iota_{D,*} \{ \sigma_D^* \alpha \cup (-\xi + \sigma_D^* c_1(S)) \} \cup \iota_{D,*} \gamma \\ &= \int_D \iota_D^* \iota_{D,*} \{ \sigma_D^* \alpha \cup (-\xi + \sigma_D^* c_1(S)) \} \cup \gamma \\ &= \int_D \sigma_D^* \alpha \cup (-\xi + \sigma_D^* c_1(S)) \cup [D]_{|D} \cup \gamma \end{aligned}$$

$$\begin{aligned}
&= \int_D \sigma_D^* \alpha \cup (-\xi + \sigma_D^* c_1(S)) \cup \xi \cup \gamma \\
&= \int_D \sigma_D^* \alpha \cup (-\xi^2 + \sigma_D^* c_1(S) \xi) \cup \gamma \\
&= \int_D \sigma_D^* \alpha \cup \sigma_D^* c_2(S) \cup \gamma \\
&= \int_D \sigma_D^* (\alpha \cup [\Delta]_{|\Delta}) \cup \gamma \\
&= \int_D \sigma_D^* \iota_{\Delta}^* (\iota_{\Delta, *}\alpha) \cup \gamma \\
&= \int_{\widetilde{S \times S}} \sigma^* \iota_{\Delta, *}\alpha \cup \iota_{D, *}\gamma
\end{aligned}$$

and this shows (3.6). \square

3.1.2 The Hodge conjecture for $S \times S$ is equivalent to the Hodge conjecture for $\text{Hilb}^2(S)$

Lemma 3.1.2.1. *Let (W, q) be a \mathbb{Q} -vector space with a non-degenerate, symmetric bilinear form. The isomorphism $\gamma : W \otimes_{\mathbb{Q}} W \xrightarrow{\sim} \text{End}_{\mathbb{Q}}(W)$ induced by q identifies q -self-adjoint endomorphisms with symmetric tensors.*

Proof. The symmetric group \mathfrak{S}_2 acts on $W^{\otimes 2}$ by exchanging the factors and on $\text{End}_{\mathbb{Q}} W$ by adjunction with respect to q . Let $v_i, w_i \in W$, let $\tau \in \mathfrak{S}_2$ be the generator. By definition of γ

$$v := \sum_i v_i \otimes w_i \xrightarrow{\gamma} \{u \mapsto \sum_i q(v_i, u) w_i\}.$$

Then clearly for all $u, r \in W$

$$q(\gamma(v)(u), r) = q(u, \gamma(\tau(v))(r))$$

which shows that v is \mathfrak{S}_2 -invariant if and only if $\gamma(v)$ is q -self-adjoint. \square

Proposition 3. *Let S be a K3 surface. Then the Hodge conjecture is true for $S \times S$ if and only if it is true for $\text{Hilb}^2(S)$.*

Proof. The Hodge conjecture for $S \times S$ implies the Hodge conjecture for $\text{Hilb}^2(S)$ because we have a dominant rational map $S \times S \dashrightarrow \text{Hilb}^2(S)$.

Conversely, assume that the Hodge conjecture is true for $\text{Hilb}^2(S)$. Following the discussion in Section 1.1.2, the space of interesting Hodge classes on $S \times S$ is identified with $E(S) := \text{End}_{\text{Hdg}}(T(S))$ by using the quadratic form q on $H^2(S, \mathbb{Q})$ given by the cup-product. If the field $E(S)$ is a CM field, then Mukai's Corollary 1.1.4.2 implies that the Hodge conjecture is true for $S \times S$. Hence we may assume without loss of generality, that S has real multiplication. Now by Zarhin's Theorem 1.1.3.1,

this means that $E(S)$ contains only q -self-adjoint endomorphisms of $T(S)$. By Lemma 3.1.2.1 such endomorphisms correspond to symmetric tensors in $T(S) \otimes T(S) \subset H^2(S, \mathbb{Q}) \otimes H^2(S, \mathbb{Q})$. It is clearly enough to show that the pullback of any such class becomes algebraic on $\widetilde{S \times S}$. But (3.1) implies that $\sigma^*(H^2(S, \mathbb{Q}) \otimes H^2(S, \mathbb{Q}))^{\mathfrak{S}_2} \subset \pi^*H^4(\text{Hilb}^2(S), \mathbb{Q})$ and the algebraicity of Hodge classes in the latter space is guaranteed by the hypothesis on $\text{Hilb}^2(S)$. \square

Remark. Here we use Mukai's theorem for all possible Picard numbers. Since the cases $\rho(S) \leq 4$ have no published proof yet, our result is waterproof only for $\rho(S) \geq 5$.

3.2 Tautological bundles on the Hilbert square

We keep the notation of the preceding section, with the only exception that S denotes an arbitrary smooth, projective surface (i.e. we don't assume $H^1(S) = 0$). Let $Z \subset S \times \text{Hilb}^2(S)$ be the universal subscheme, denote by $p : S \times \text{Hilb}^2(S) \rightarrow S$ and by $q : S \times \text{Hilb}^2(S) \rightarrow \text{Hilb}^2(S)$ the projections. Since $q|_Z$ is flat and finite of degree 2, for any line bundle $\mathcal{L} \in \text{Pic}(S)$ the coherent sheaf

$$\mathcal{L}^{[2]} := q_{|Z,*}(p^*\mathcal{L}|_Z)$$

is a rank two vector bundle on $\text{Hilb}^2(S)$, called the *tautological bundle associated with \mathcal{L}* . We will now calculate the Chern character of $\mathcal{L}^{[2]}$ and show that it is a stable vector bundle for an appropriate choice of a polarization if $h^0(\mathcal{L}) \geq 2$.

3.2.1 The fundamental short exact sequence

Recall that $D \subset \widetilde{S \times S}$ is the exceptional divisor of the blowup $\sigma : \widetilde{S \times S} \rightarrow S \times S$. The following lemma is well-known and can be found for example in the proof of [Da, Prop. 2.3]. For the convenience of the reader, we recall the argument.

Lemma 3.2.1.1. *On $\widetilde{S \times S}$ there is a short exact sequence*

$$0 \rightarrow \pi^*\mathcal{L}^{[2]} \rightarrow \widetilde{r}_1^*\mathcal{L} \oplus \widetilde{r}_2^*\mathcal{L} \rightarrow \mathcal{L}_D \rightarrow 0$$

where $\mathcal{L}_D = \iota_{D,*}\sigma_D^*\mathcal{L}$.

Proof. Let $\Delta_{ij} := \{(s_1, s_2, s_3) \in S^3 \mid s_i = s_j\} \subset S \times S \times S$. Then the pullback $\widetilde{Z} := (\text{id} \times \pi)^{-1}(Z) \subset S \times \widetilde{S \times S}$ decomposes as the scheme-theoretic union $\widetilde{Z} = \widetilde{Z}_1 \cup \widetilde{Z}_2$ with $\widetilde{Z}_1 = (\text{id} \times \sigma)^{-1}(\Delta_{12})$ and $\widetilde{Z}_2 = (\text{id} \times \sigma)^{-1}(\Delta_{13})$. This is visualized by the commutative diagram

$$\begin{array}{ccccc}
 \Delta_{12} \cup \Delta_{13} & \longleftarrow & \widetilde{Z} = \widetilde{Z}_1 \cup \widetilde{Z}_2 & \longrightarrow & Z \\
 \downarrow & & \downarrow & & \downarrow \\
 S \times S \times S & \xleftarrow{\text{id} \times \sigma} & S \times \widetilde{S \times S} & \xrightarrow{\text{id} \times \pi} & S \times \text{Hilb}^2(S) \\
 \downarrow \text{pr}_2 \times \text{pr}_3 & & \downarrow \widetilde{q} & \searrow \widetilde{p} & \downarrow q \\
 & & & S & \\
 & & & \swarrow p & \\
 S \times S & \xleftarrow{\sigma} & \widetilde{S \times S} & \xrightarrow{\pi} & \text{Hilb}^2(S).
 \end{array} \tag{3.7}$$

Note that $\tilde{q}_{|\tilde{Z}_i} : \tilde{Z}_i \rightarrow \widetilde{S \times S}$ is an isomorphism and that the diagram

$$\begin{array}{ccc}
 \tilde{Z}_i & \xrightarrow{\tilde{q}_{|\tilde{Z}_i}} & \widetilde{S \times S} \\
 \tilde{p}_{|\tilde{Z}_i} \searrow & & \nearrow \tilde{r}_i \\
 & S &
 \end{array} \tag{3.8}$$

is commutative.

Let $\tilde{D} = \tilde{Z}_1 \cap \tilde{Z}_2$, such that $\tilde{D} = (\text{id} \times \sigma)^{-1}(\Delta_{123})$ where $\Delta_{123} = \{(s_1, s_2, s_3) \in S^3 \mid s_1 = s_2 = s_3\}$. Then $\tilde{q}_{|\tilde{D}} : \tilde{D} \rightarrow D$ is an isomorphism which makes the diagram

$$\begin{array}{ccc}
 \tilde{D} & \xrightarrow{\tilde{q}_{|\tilde{D}}} & D \\
 \tilde{p}_{|\tilde{D}} \searrow & & \nearrow \sigma_D \\
 & S &
 \end{array} \tag{3.9}$$

commutative. Now, tensor the short exact sequence

$$0 \rightarrow \mathcal{O}_{\tilde{Z}} \rightarrow \mathcal{O}_{\tilde{Z}_1} \oplus \mathcal{O}_{\tilde{Z}_2} \rightarrow \mathcal{O}_{\tilde{D}} \rightarrow 0 \tag{3.10}$$

on $S \times \widetilde{S \times S}$ with $\tilde{p}^* \mathcal{L}$ and consider its push-forward under \tilde{q} . By the diagrams (3.8) and (3.9) we get

$$\tilde{q}_*(\tilde{p}^* \mathcal{L}_{|\tilde{Z}_i}) \simeq \tilde{r}_i^* \mathcal{L} \quad \text{and} \quad \tilde{q}_*(\tilde{p}^* \mathcal{L}_{|\tilde{D}}) \simeq \mathcal{L}_D.$$

Affine base change (q_Z is finite, hence affine) applied to the Cartesian square

$$\begin{array}{ccc}
 \tilde{Z} & \longrightarrow & Z \\
 \tilde{q}_{|\tilde{Z}} \downarrow & & \downarrow \\
 \widetilde{S \times S} & \xrightarrow{\pi} & \text{Hilb}^2(S)
 \end{array}$$

implies that $\tilde{q}_*(\tilde{p}^* \mathcal{L}_{|\tilde{Z}}) \simeq \pi^* \mathcal{L}^{[2]}$. Since $\tilde{q}_{|\tilde{Z}}$ is affine, we get $R^1 \tilde{q}_*(\tilde{p}^* \mathcal{L}_{|\tilde{Z}}) = 0$ and therefore the push-forward of (3.10) tensored with $\tilde{p}^* \mathcal{L}$ is the short exact sequence we are looking for. \square

3.2.2 The Chern character of $\mathcal{L}^{[2]}$

The lemma shows that

$$\pi^* \text{ch}(\mathcal{L}^{[2]}) = \tilde{r}_1^* \text{ch}(\mathcal{L}) + \tilde{r}_2^* \text{ch}(\mathcal{L}) - \text{ch}(\mathcal{L}_D). \tag{3.11}$$

The Grothendieck–Riemann–Roch theorem gives

$$\begin{aligned}
\mathrm{ch}(\mathcal{L}_D) &= \iota_{D,*} \left(\mathrm{ch}(\sigma_D^* \mathcal{L}) \mathrm{td}(\mathcal{N}_{D|\widetilde{S \times S}})^{-1} \right) \\
&= \iota_{D,*} \left(\sigma_D^* \mathrm{ch}(\mathcal{L}) \mathrm{td}(\mathcal{O}_D(D))^{-1} \right) \\
&= \iota_{D,*} \left(1 + \sigma_D^* c_1(\mathcal{L}) + \dots \right) \left(1 - \frac{[D]}{2} + \frac{[D]^2}{6} + \dots \right) \\
&= \left([D] + \frac{\widetilde{r}_1^* c_1(\mathcal{L}) + \widetilde{r}_2^* c_1(\mathcal{L})}{2} [D] + \dots \right) \left(1 - \frac{[D]}{2} + \frac{[D]^2}{6} + \dots \right) \\
&= [D] + \frac{-[D]^2 + (\widetilde{r}_1^* c_1(\mathcal{L}) + \widetilde{r}_2^* c_1(\mathcal{L})) [D]}{2} \\
&\quad + \text{terms of coh. degree} \geq 6.
\end{aligned} \tag{3.12}$$

Combining (3.11) with (3.12) yields

$$\begin{aligned}
\pi^* \mathrm{ch}(\mathcal{L}^{[2]}) &= 2 + \widetilde{r}_1^* c_1(\mathcal{L}) + \widetilde{r}_2^* c_1(\mathcal{L}) - [D] \\
&\quad + \frac{\widetilde{r}_1^* c_1^2(\mathcal{L}) + \widetilde{r}_2^* c_1^2(\mathcal{L}) - (\widetilde{r}_1^* c_1(\mathcal{L}) + \widetilde{r}_2^* c_1(\mathcal{L})) [D] + [D]^2}{2} \\
&\quad + \text{terms of coh. degree} \geq 6 \\
&= 2 + \widetilde{r}_1^* c_1(\mathcal{L}) + \widetilde{r}_2^* c_1(\mathcal{L}) - [D] \\
&\quad + \frac{q(c_1(\mathcal{L})) (\sigma^* [\Delta]^{4,0} + \sigma^* [\Delta]^{0,4})}{2} \\
&\quad - \frac{(\widetilde{r}_1^* c_1(\mathcal{L}) + \widetilde{r}_2^* c_1(\mathcal{L})) [D] - [D]^2}{2} \\
&\quad + \text{terms of coh. degree} \geq 6.
\end{aligned} \tag{3.13}$$

Here $[\Delta]^{i,j}$ refers to the (i, j) -th Künneth factor of the diagonal in $S \times S$ and q is the intersection product on S .

Lemma 3.2.2.1. *Let S be a K3 surface. Then*

$$\begin{aligned}
\pi^* c_2(X) &= \sigma^* c_2(S \times S) - 3[D]^2 \\
&= 24\sigma^*([\Delta]^{4,0} + [\Delta]^{0,4}) - 3\pi^* \delta^2.
\end{aligned}$$

Here, $\delta \in H^2(X, \mathbb{Q})$ is the class with $\pi^*(\delta) = [D]$.

Proof. Let $E \subset X$ be the exceptional divisor of the Hilbert–Chow morphism $X \rightarrow \mathrm{Sym}^2(S)$. Then E is smooth and this is precisely the branch locus of $\pi : \widetilde{S \times S} \rightarrow X$. Let $s \in H^0(X, \mathcal{O}_X(E))$ be the defining section of E . Then there exists a line bundle \mathcal{A} on X with the properties (see e.g. [Mum]):

- $\pi_* \mathcal{O}_{\widetilde{S \times S}} \simeq \mathcal{O}_X \oplus \mathcal{A}^{-1}$ and \mathcal{A}^{-1} is the eigenspace to the eigenvalue -1 of the cover involution of π .
- $\mathcal{A}^{\otimes 2} \simeq \mathcal{O}(E)$,

• Let $\rho: |\mathcal{A}| \rightarrow X$ be the projection of the total space of \mathcal{A} to X and let $t \in H^0(|\mathcal{A}|, \rho^*\mathcal{A})$ be the tautological section. Then $\widetilde{S \times S} \rightarrow X$ is isomorphic to $Z(\rho^*s - t^{\otimes 2}) \subset |\mathcal{A}|$ with the restriction of the projection to X .

It follows that if s, z_2, z_3, z_4 are local coordinates on X , then $t, \tilde{z}_2 := \rho^*z_2, \tilde{z}_3 := \rho^*z_3, \tilde{z}_4 := \rho^*z_4$ are local coordinates on $\widetilde{S \times S}$ and π is given by

$$(t, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4) \xrightarrow{\pi} (t^2, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4). \quad (3.14)$$

The ramification divisor R of π is by definition the zero locus of the pullback morphism $\pi^*\omega_X \rightarrow \omega_{\widetilde{S \times S}}$. Thus $\mathcal{O}(R) \simeq \omega_{\widetilde{S \times S}} \otimes \pi^*\omega_X^{-1} \simeq \omega_{\widetilde{S \times S}} \simeq \mathcal{O}(D)$. On the other hand, (3.14) shows that in local coordinates as above $\pi^*dz_i = d\tilde{z}_i$ and $\pi^*ds = dt^2 = 2tdt$. Thus, $R = Z(t)$. Since $t \in H^0(\widetilde{S \times S}, \pi^*\mathcal{A})$, this shows that $\pi^*\mathcal{A} \simeq \mathcal{O}(D)$.

On $\widetilde{S \times S}$ we have a short exact sequence

$$0 \longrightarrow \mathcal{T}_{\widetilde{S \times S}} \xrightarrow{d\pi} \pi^*\mathcal{T}_X \xrightarrow{\psi} \mathcal{O}_D(2D) \longrightarrow 0 \quad (3.15)$$

This is of course well-known, but we didn't find a proof in the literature and therefore we give the argument. We have $\pi^*\mathcal{O}_E(E) \simeq \mathcal{O}_D(2D)$. The surjection $\psi: \pi^*\mathcal{T}_X \rightarrow \mathcal{O}_D(2D)$ is defined as the pullback of the composition $\mathcal{T}_X \rightarrow \mathcal{T}_{X|E} \rightarrow \mathcal{N}_{E/X} \simeq \mathcal{O}_E(E)$.

Now we check fibre by fibre that (3.15) is exact. Outside of D , the differential $d\pi$ is clearly an isomorphism, so let $y \in D$. Then the local description of π in (3.14) shows that

$$\begin{aligned} d\pi \left(\frac{\partial}{\partial t|_y} \right) &= 2t(y) \frac{\partial}{\partial s|_{\pi(y)}} = 0 \text{ and} \\ d\pi \left(\frac{\partial}{\partial \tilde{z}_i|_y} \right) &= \frac{\partial}{\partial z_i|_{\pi(y)}} \text{ for } i \geq 2. \end{aligned}$$

Since the kernel of $\psi(y)$ is spanned by the $\left(\frac{\partial}{\partial z_i|_{\pi(y)}} \right)_{i=2,3,4}$, the sequence (3.15) is exact.

It follows from (3.15) that

$$\text{ch}(\pi^*\mathcal{T}_X) = \text{ch}(\mathcal{T}_{\widetilde{S \times S}}) + \text{ch}(\mathcal{O}_D(2D)). \quad (3.16)$$

We use the formulas in [Fu], Example 15.4.3 to deduce

$$c_1(\mathcal{T}_{\widetilde{S \times S}}) = -[D] \text{ and } c_2(\mathcal{T}_{\widetilde{S \times S}}) = \sigma^*c_2(S \times S) - [D]^2.$$

Hence

$$\text{ch}(\mathcal{T}_{\widetilde{S \times S}}) = 4 - [D] - \sigma^*c_2(S \times S) + \frac{3}{2}[D]^2 + \text{terms of degree } \geq 6. \quad (3.17)$$

Next, by Grothendieck–Riemann–Roch as above we have

$$\mathrm{ch}(\mathcal{O}_D(2D)) = [D] + \frac{3[D]^2}{2} + \text{terms of degree } \geq 6. \quad (3.18)$$

Now combine (3.17) with (3.18) to see that

$$\pi^* \mathrm{ch}(\mathcal{T}_X) = 4 + 3[D]^2 - \sigma^* c_2(S \times S) + \text{terms of degree } \geq 6.$$

This proves the first equality. The second one follows from the equality $c_2(S \times S) = r_1^* c_2(S) + r_2^* c_2(S)$ (use that $c_1(S) = 0$) and from $c_2(S) = 24[x]$ for an arbitrary point $x \in S$. \square

Corollary 3.2.2. *Let S be a K3 surface and let $\mathcal{L} \in \mathrm{Pic}(S)$. The Chern character of $\mathcal{L}^{[2]}$ is*

$$\begin{aligned} \mathrm{ch}(\mathcal{L}^{[2]}) &= 2 + \{\varphi(c_1(\mathcal{L})) - \delta\} \\ &\quad + \left\{ \frac{q(c_1(\mathcal{L}))}{48} c_2(X) - \frac{1}{2} \varphi(c_1(\mathcal{L})) \delta + \left(\frac{q(c_1(\mathcal{L})) + 8}{16} \right) \delta^2 \right\} \\ &\quad + \text{terms of coh. degree } \geq 6, \end{aligned}$$

where q is the intersection product on S .

3.2.3 The stability of $\mathcal{L}^{[2]}$

Let H be an ample divisor on a smooth projective variety Y , let \mathcal{E} be a torsion-free coherent sheaf on Y . The *slope of \mathcal{E} with respect to H* is defined as

$$\mu_H(\mathcal{E}) := \frac{\int_Y c_1(\mathcal{E})[H]^{\dim(Y)-1}}{\mathrm{rk}(\mathcal{E})}. \quad (3.19)$$

A torsion-free coherent sheaf \mathcal{E} on X is called μ_H -*stable* if for any subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$ we have

$$\mu_H(\mathcal{F}) < \mu_H(\mathcal{E}).$$

Let now S be an arbitrary smooth, projective surface, let H be an ample divisor on S . Then for $N \gg 0$ the divisor $\widetilde{H}_N := \widetilde{r}_1^*(NH) + \widetilde{r}_2^*(NH) - D$ is ample on $\widetilde{S \times S}$ and it is of the form $\pi^* \overline{H}_N$ for an ample divisor \overline{H}_N on $X = \mathrm{Hilb}^2(S)$.

Proposition 3.2.3.1. *Let $\mathcal{L} \in \mathrm{Pic}(S)$ be a line bundle with $h^0(\mathcal{L}) \geq 2$. Then for $N \gg 0$, the vector bundle $\mathcal{L}^{[2]}$ is $\mu_{\overline{H}_N}$ -stable on $\mathrm{Hilb}^2(S)$.*

Proof. If $\mathcal{L}^{[2]}$ had a destabilizing subsheaf, then by passing to the reflexive hull we see that there would exist a destabilizing line bundle. Thus, any

destabilizing subsheaf of $\mathcal{L}^{[2]}$ on X with respect to \overline{H}_N induces a destabilizing sub-line bundle of $\mathcal{E} := \pi^* \mathcal{L}^{[2]}$ on $\widetilde{S} \times \widetilde{S}$ with respect to \widetilde{H}_N . We will show that \mathcal{E} is $\mu_{\widetilde{H}_N}$ -stable. This will finish the proof of the proposition.

For $i = 1, 2$, put $\mathcal{L}_i := \widetilde{r}_i^* \mathcal{L}_i$. We have seen in Lemma 3.2.1.1 that there exists a short exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \mathcal{L}_D \rightarrow 0 \quad (3.20)$$

where the surjection on the right is given by $(s_1, s_2) \mapsto s_{1|D} - s_{2|D}$. We see from this that \mathcal{E} contains the line bundles $\mathcal{L}_1(-D)$ and $\mathcal{L}_2(-D)$.

Let now $\mathcal{A} \subset \mathcal{E}$ be an arbitrary sub-line bundle. Then \mathcal{A} has one of the following three properties:

- 1.) $\mathcal{A} \subset \mathcal{L}_1(-D)$,
- 2.) $\mathcal{A} \subset \mathcal{L}_2(-D)$,
- 3.) $\mathcal{A} \not\subset \mathcal{L}_1(-D)$ and $\mathcal{A} \not\subset \mathcal{L}_2(-D)$.

We will prove that there exist $N_1, N_2, N_3 \in \mathbb{N}$ such that for all $\mathcal{A} \subset \mathcal{E}$ with property i we have

$$\mu_{\widetilde{H}_N}(\mathcal{A}) < \mu_{\widetilde{H}_N}(\mathcal{E}) \quad (3.21)$$

for all $N \geq N_i$.

Assume first that we are in case 1.), i.e. that $\mathcal{A} \subset \mathcal{L}_1(-D)$. Choose a natural number $N_1 \geq 4$ such that \widetilde{H}_N is ample for all $N \geq N_1$. Then $\mu_{\widetilde{H}_N}(\mathcal{A}) \leq \mu_{\widetilde{H}_N}(\mathcal{L}_1(-D))$ for all $N \geq N_1$. Let $\alpha_i := c_1(\mathcal{L}_i)$. Then $c_1(\mathcal{E}) = \alpha_1 + \alpha_2 - [D]$ and therefore

$$\begin{aligned} \mu_{\widetilde{H}_N}(\mathcal{E}) - \mu_{\widetilde{H}_N}(\mathcal{A}) &\geq \mu_{\widetilde{H}_N}(\mathcal{E}) - \mu_{\widetilde{H}_N}(\mathcal{L}_1(-D)) \\ &= \int_{\widetilde{S} \times \widetilde{S}} \left(\frac{\alpha_1 + \alpha_2 - [D]}{2} - (\alpha_1 - [D]) \right) [\widetilde{H}_N]^3 \\ &= \int_{\widetilde{S} \times \widetilde{S}} \frac{\alpha_2 - \alpha_1}{2} [\widetilde{H}_N]^3 + \int_{\widetilde{S} \times \widetilde{S}} \frac{[D]}{2} [\widetilde{H}_N]^3 \\ &= 0 + \int_{\widetilde{S} \times \widetilde{S}} \frac{[D]}{2} [\widetilde{H}_N]^3 \\ &> 0 \end{aligned} \quad (3.22)$$

for all $N \geq N_1$.

An analogous reasoning applies in case 2.) with $N_2 = N_1$.

In case 3.) we proceed in two steps. First we show that

$$\mu_F(\mathcal{A}) < \mu_F(\mathcal{E}), \quad (3.23)$$

where $F = \widetilde{r}_1^* H + \widetilde{r}_2^* H$ and μ_F is the slope with respect to the nef divisor F , defined as in (3.19). Then we use an asymptotic argument to complete the proof.

To prove (3.23), we will consider two divisors in $|MF|$ which intersect along a reducible surface. Then we reduce our computation to the irreducible components of these surfaces.

Choose M sufficiently positive such that the linear system $|MH|$ contains two distinct, smooth curves C and C' meeting transversely. Denote by

$$G_1 := (C \times S) \cup (S \times C') \quad \text{and by} \quad G_2 := (C' \times S) \cup (S \times C),$$

let \widetilde{G}_1 and \widetilde{G}_2 be their strict transforms under $\sigma : \widetilde{S \times S} \rightarrow S \times S$. Note that $\widetilde{G}_i \in |MF|$ because each component of G_i meets the center of the blowup, namely the diagonal of $S \times S$, along a curve. The intersection $\widetilde{G}_1 \cap \widetilde{G}_2$ is the disjoint union of the four smooth surfaces

$$\begin{aligned} T_1 &:= (C \cap C') \times S, & T_2 &:= S \times (C \cap C'), \\ T_3 &:= \widetilde{C \times C}, & T_4 &:= \widetilde{C' \times C'}, \end{aligned}$$

where for any subvariety $Y \subsetneq S \times S$ we write \widetilde{Y} for the strict transform of Y under $\sigma : \widetilde{S \times S} \rightarrow S \times S$.

Then for the F -slope of any coherent sheaf \mathcal{F} we find

$$\mu_{MF}(\mathcal{F}) = \frac{\deg_{MF}(\mathcal{F})}{\text{rk}(\mathcal{F})} = \sum_{i=1}^4 \frac{\deg_{T_i}(\mathcal{F})}{\text{rk}(\mathcal{F})}$$

where $\deg_{T_i}(\mathcal{F}) := \int_{T_i} c_1(\mathcal{F}|_{T_i}) \cup [MF]|_{T_i}$. We will show that

$$\deg_{T_i}(\mathcal{A}) \leq \frac{\deg_{T_i}(\mathcal{E})}{2}$$

for $i = 1, \dots, 4$ with strict inequality for $i = 3, 4$. This will conclude the proof of (3.23).

$i = 1$: The surface T_1 is a disjoint union of surfaces of the form $S_p := \widetilde{\{p\} \times S}$, $p \in S$ running over the finite set $C \cap C'$. Note that S_p is isomorphic to the blow-up of S in p . Denote by $\sigma_p : S_p \rightarrow S$ the blow-down and by $E \subset S_p$ the exceptional divisor. Then

$$c_1((\mathcal{E})|_{S_p}) = \sigma_p^* c_1(\mathcal{L}) - [E]$$

because $c_1(\mathcal{E}) = \widetilde{r}_1^* c_1(\mathcal{L}) + \widetilde{r}_2^* c_1(\mathcal{L}) - [D]$ and because $\widetilde{r}_1^* \mathcal{L}|_{S_p} = \mathcal{O}_{S_p}$. Now suppose that

$$2 \deg_{T_1}(\mathcal{A}) > \deg_{T_1}(\pi^* \mathcal{L}^{[2]}). \quad (3.24)$$

Then we would get

$$2 \int_{S_p} c_1(\mathcal{A}|_{S_p}) [MF]|_{S_p} > \int_{S_p} (\sigma_p^* c_1(\mathcal{L}) - [E]) [MF]|_{S_p} \geq 0$$

because $[MF]_{|S_p}$ is a nef class on S_p and because $\sigma_p^*\mathcal{L} \otimes \mathcal{O}(-E)$ is the line bundle of an effective divisor on S_p . Indeed, since $h^0(\mathcal{L}) \geq 2$, there exists a divisor $K \in |\mathcal{L}|$ with $p \in \text{supp}(K)$. Then the strict transform \tilde{K} of K is in the linear system $|\sigma_p^*\mathcal{L} \otimes \mathcal{O}(-kE)|$ for some $k \geq 1$. Thus, $\tilde{K} + (k-1)E$ is an effective divisor with line bundle $\sigma_p^*\mathcal{L} \otimes \mathcal{O}(-E)$.

By (3.20), $\mathcal{A}_{|S_p} \subset (\mathcal{L}_1 \oplus \mathcal{L}_2)_{|S_p} = \mathcal{O}_{S_p} \oplus \sigma_p^*\mathcal{L}$. Now, since $\mathcal{A}_{|S_p}$ has positive slope, the composition $\mathcal{A}_{|S_p} \rightarrow \mathcal{O}_{S_p} \oplus \sigma_p^*\mathcal{L} \rightarrow \mathcal{O}_{S_p}$ must be zero. Under the assumption (3.24), this composition is zero for all $p \in S$ because all surfaces S_p for $p \in S$ have the same fundamental class in $\widetilde{S \times S}$. On the other hand, all $x \in \widetilde{S \times S}$ lie on some S_p . This shows that the composition

$$\mathcal{A} \rightarrow \mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \mathcal{L}_1$$

is zero. But then the short exact sequence (3.20) implies that $\mathcal{A} \subset \tilde{r}_2^*\mathcal{L}(-D)$, because the surjection $\mathcal{L}_1 \oplus \mathcal{L}_2 \rightarrow \mathcal{L}_D$ is given by $(s_1, s_2) \mapsto s_{1|D} - s_{2|D}$. This is a contradiction to the assumption that \mathcal{A} satisfies 3.).

$i = 2$: analogous to $i = 1$.

$i = 3$: Note that $\widetilde{C \times C}$ is isomorphic to $C \times C$ and that

$$MF_{|\widetilde{C \times C}} = M(p_1^*H_{|C} + p_2^*H_{|C})$$

where $p_i : C \times C \rightarrow C$ are the projections. Moreover, it is easily checked that

$$(\pi^*\mathcal{L}^{[2]})_{|\widetilde{C \times C}} = \pi_C^*\mathcal{L}_{|C}^{[2]}$$

where $\pi_C : C \times C \rightarrow \text{Hilb}^2(C)$ is the natural projection and $\mathcal{L}_{|C}^{[2]}$ is the tautological line bundle associated with $\mathcal{L}_{|C}$ on $\text{Hilb}^2(C)$. It remains to apply [Mi], Cor. 4.3.3 which says that $\pi_C^*\mathcal{L}_{|C}^{[2]}$ is a stable vector bundle on $C \times C$.

$i = 4$ analogous to $i = 3$.

Thus, we have proved (3.23). To conclude the proof of the proposition we note that for $n \in \mathbb{N}$ we have $\tilde{H}_{n+N_1} = nF + \tilde{H}_{N_1}$. Define the function

$$\begin{aligned} \varphi_n : K^0(\text{Coh}(\widetilde{S \times S})) &\rightarrow \mathbb{Q} \\ \mathcal{F} &\mapsto \sum_{i=0}^2 n^i \frac{3!}{i!(3-i)!} \int_{\widetilde{S \times S}} c_1(\mathcal{F})[F]^i [\tilde{H}_{N_1}]^{3-i} \end{aligned}$$

Then for all $\mathcal{F} \in \text{Coh}(\widetilde{S \times S})$ one has

$$\mu_{\tilde{H}_{n+N_1}}(\mathcal{F}) = n^3 \mu_F(\mathcal{F}) + \frac{\varphi_n(\mathcal{F})}{\text{rk}(\mathcal{F})} \quad (3.25)$$

Inequality (3.23) implies that there exists a positive constant $k > 0$ such that for all sub-line bundles $\mathcal{A} \subset \mathcal{E}$ with property 3.) we have

$$\mu_F(\mathcal{E}) - \mu_F(\mathcal{A}) \geq k. \quad (3.26)$$

This is because μ_F takes integer values on line bundles.

We will now show that $\varphi_n(\mathcal{A}) < \varphi_n(\mathcal{E})$. To see this, note first that $\mathcal{A} \cap \mathcal{L}_1(-D) = 0$. Indeed, otherwise the torsion-free sheaf $\mathcal{A} + \mathcal{L}_1(-D)$ would be of rank 1 as follows from the short exact sequence

$$0 \rightarrow \mathcal{A} \cap \mathcal{L}_1(-D) \rightarrow \mathcal{A} \oplus \mathcal{L}_1(-D) \rightarrow \mathcal{A} + \mathcal{L}_1(-D) \rightarrow 0.$$

Then the reflexive hull \mathcal{A}' of $\mathcal{A} + \mathcal{L}_1(-D)$ would be a sub-line bundle of \mathcal{E} which again would have property 3.) because $\mathcal{A} \subset \mathcal{A}'$. From what we have seen above we deduce $\mu_F(\mathcal{A}') < \mu_F(\mathcal{E})$. On the other hand

$$\begin{aligned} \mu_F(\mathcal{L}_1(-D)) &= \int_{\widetilde{S \times S}} (\alpha_1 - [D])[F]^3 \\ &= \int_{\widetilde{S \times S}} \alpha_1 [F]^3 \\ &= \int_{\widetilde{S \times S}} \frac{\alpha_1 + \alpha_2}{2} [F]^3 \\ &= \int_{\widetilde{S \times S}} \frac{\alpha_1 + \alpha_2 - [D]}{2} [F]^3 \\ &= \mu_F(\mathcal{E}). \end{aligned}$$

Here, we used that the integral over a cohomology class of degree 6 on D which is pulled back from Δ vanishes. Using that $\mathcal{L}_1(-D) \subset \mathcal{A}'$ and that F is nef, we get a chain of inequalities

$$\mu_F(\mathcal{E}) = \mu_F(\mathcal{L}_1(-D)) \leq \mu_F(\mathcal{A}') < \mu_F(\mathcal{E}).$$

This is a contradiction, whence $\mathcal{A} \cap \mathcal{L}_1(-D) = 0$. Then we have a short exact sequence

$$0 \rightarrow \mathcal{A} \oplus \mathcal{L}_1(-D) \rightarrow \mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0 \quad (3.27)$$

where \mathcal{Q} is a torsion sheaf. It follows that $c_1(\mathcal{Q})$ is either zero or effective. Since φ_n involves only products of the nef divisor F and the ample divisor \widetilde{H}_{N_1} , this implies that $\varphi_n(\mathcal{Q}) \geq 0$. We claim that there exists $n_1 \in \mathbb{N}$ such that for $n \geq n_1$ we have $\varphi_n(\mathcal{L}_1(-D)) > 0$. To see this, it is enough to show that the n^2 -term of $\varphi_n(\mathcal{L}_1(-D))$ is positive. We have

$$\begin{aligned} c_1(\mathcal{L}_1(-D))[F]^2[\widetilde{H}_{N_1}] &= (\alpha_1 - [D])[F]^2(N_1[F] - [D]) \\ &= N_1\alpha_1[F]^3 + [F]^2[D]^2 - [D](N_1[F]^3 + \alpha_1[F]^2). \end{aligned}$$

If q denotes the intersection product on S , then we obtain

$$\begin{aligned}
\int_{\widetilde{S \times S}} N_1 \alpha_1 [F]^3 + \int_{\widetilde{S \times S}} [F]^2 [D]^2 &= N_1 \int_{S \times S} r_1^* c_1(\mathcal{L}_1) (r_1^*[H] + r_2^*[H])^3 \\
&\quad + \int_D [F]^2 \xi \\
&= \left(3N_1 q(c_1(\mathcal{L}_1), [H]) - 4 \right) q([H]) \\
&> 0.
\end{aligned}$$

Here, $r_i : S \times S \rightarrow S$ are the projections and $\xi = [D]_{|D} = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{N}_{\Delta|S \times S})}(-1))$. In the last inequality we use that $N_1 \geq 4$ and that $c_1(\mathcal{L})$ is an effective class on S . This proves the existence of n_1 with $\varphi_n(\mathcal{L}_1(-D)) > 0$ for all $n \geq n_1$. Now by (3.27) we have

$$\varphi_n(\mathcal{A}) = \varphi_n(\mathcal{E}) - \varphi_n(\mathcal{Q}) - \varphi_n(\mathcal{L}_1(-D)) < \varphi_n(\mathcal{E}) \quad (3.28)$$

for all $n \geq n_1$. Putting together (3.25), (3.26) and (3.28) we find for $n \geq n_1$ and for all line bundles $\mathcal{A} \subset \mathcal{E}$ with property 3.)

$$\begin{aligned}
\mu_{\tilde{H}_{n+N_1}}(\mathcal{E}) - \mu_{\tilde{H}_{n+N_1}}(\mathcal{A}) &= n^3(\mu_F(\mathcal{E}) - \mu_F(\mathcal{A})) + \frac{\varphi_n(\mathcal{E})}{2} - \varphi_n(\mathcal{A}) \\
&> n^3 k - \frac{\varphi_n(\mathcal{E})}{2}.
\end{aligned}$$

Now since $k > 0$ and since $\varphi_n(\mathcal{E})$ is a polynomial of degree 2 in n , there exists $n_2 \geq n_1$ such that $n^3 k - \frac{\varphi_n(\mathcal{E})}{2} > 0$ for all $n \geq n_2$. Therefore, with $N_3 := N_1 + n_2$, inequality (3.21) is satisfied for $i = 3$. This completes the proof of the proposition. \square

3.3 The Fano variety of lines on a cubic fourfold

In this section we study cubic fourfolds and its varieties of lines. These Fano varieties are known to be deformations of the second Hilbert scheme of a K3 surface (see Theorem 3.3.1.1 below) and sometimes they are even isomorphic to such a Hilbert scheme. We calculate fundamental classes of surfaces in the Fano variety which are naturally induced by the geometry of Y . On the way we have to express the Chern classes of F in terms of Schubert cycles.

We summarize here the notation we will use in the sequel. Let

- $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold.
- $F \subset \text{Gr}(2, 6) =: \text{Gr}$ be its Fano variety of lines.
- $g \in H^2(F, \mathbb{Z})$ be the class of the Plücker polarization.
- $\mathcal{F} \hookrightarrow \mathcal{O}_F \otimes \mathbb{C}^6$ be the universal subbundle.
- $Z \subset Y \times F$ be the incidence variety, that is $Z = \{(x, [l]) \in Y \times F \mid x \in l\}$.
- $q : Z \rightarrow Y$ and $p : Z \rightarrow F$ be the restrictions of the projections.
- $\mathcal{O}_Z(-1) \hookrightarrow p^*\mathcal{F}$ be the relative tautological bundle of the \mathbb{P}^1 -bundle p .

We will need the the analogous objects for \mathbb{P}^5 and the Grassmannian Gr . Let

- $\mathcal{G} \hookrightarrow \mathcal{O}_{\text{Gr}} \otimes \mathbb{C}^6$ be the universal subbundle, \mathcal{Q} the universal quotient bundle. Thus we have an exact sequence $0 \rightarrow \mathcal{G} \rightarrow \mathcal{O}_{\text{Gr}}^{\oplus 6} \rightarrow \mathcal{Q} \rightarrow 0$.
- $\tilde{Z} \subset \mathbb{P}^5 \times \text{Gr}$ be the incidence variety.
- $\tilde{q} : \tilde{Z} \rightarrow \mathbb{P}^5$ and $\tilde{p} : \tilde{Z} \rightarrow \text{Gr}$ be the projections.
- $\mathcal{O}_{\tilde{Z}}(-1) \hookrightarrow \tilde{q}^*\mathcal{G}$ be the relative tautological bundle.
- For $[l] \in \text{Gr}$, denote by $W_{[l]}$ the corresponding 2-dimensional subspace of \mathbb{C}^6 .

3.3.1 The result of Beauville and Donagi

In [BD], Beauville and Donagi study the Fano variety of lines on a cubic fourfold Y . They obtain the following:

Theorem 3.3.1.1 (Beauville, Donagi). *Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold, let $F \subset \text{Gr}(2, 6)$ be the variety parameterizing lines on Y . Then F is a smooth, projective Hyperkähler manifold, deformation equivalent to $\text{Hilb}^2(S)$ for a K3 surface S . If $Z \subset Y \times F$ is the universal subscheme, then the correspondence*

$$\Phi : H^4(Y, \mathbb{Z}) \rightarrow H^2(F, \mathbb{Z}), \quad \alpha \mapsto p_*(q^*\alpha \cup [Z])$$

is an isomorphism of Hodge structures.

This theorem is proved by considering Pfaffian cubics which form a hypersurface in the moduli space of cubic fourfolds. To such a Pfaffian cubic

Y one can associate a K3 surface S of degree 14 in \mathbb{P}^8 and Beauville and Donagi construct explicitly an isomorphism between $F(Y)$ and $\text{Hilb}^2(S)$ if Y contains no plane and S no line.

Later, Hassett discovered that there are actually countably many hypersurfaces in the moduli space of cubic fourfolds to the members of which one can associate a K3 surface such that an isomorphism as above exists (see [Has]).

3.3.2 Chern classes of F

First, we will express $c_2(F)$ in terms of Schubert cocycles. Note that the odd Chern classes of F vanish since F is symplectic.

Lemma 3.3.2.1. *There is a canonical isomorphism*

$$\mathcal{O}_{\tilde{Z}}(-1) \simeq \tilde{q}^* \mathcal{O}_{\mathbb{P}^5}(-1).$$

Proof. The tautological bundles in question are defined by the universal inclusions

$$\mathcal{O}_{\mathbb{P}^5}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^5} \otimes \mathbb{C}^6$$

and

$$\mathcal{O}_{\tilde{Z}}(-1) \hookrightarrow \tilde{p}^* \mathcal{G} \hookrightarrow \mathcal{O}_{\tilde{Z}} \otimes \mathbb{C}^6$$

given pointwise over $x \in \mathbb{P}^5$ by $\mathbb{C}x \subset \mathbb{C}^6$ and over $(x, [l]) \in \tilde{Z}$ by $\mathbb{C}x \subset \mathcal{G}([l]) = W_{[l]} \subset \mathbb{C}^6$. The claim follows immediately. \square

Note that by restriction the analogous isomorphism exists over Z .

Lemma 3.3.2.2 (Altman, Kleiman (see [AK])). *The Fano variety $F \subset \text{Gr}$ is defined by a regular section of $\text{Sym}^3(\mathcal{G}^\vee)$.*

Proof. By Lemma 3.3.2.1, for $d \geq 0$ we get $\tilde{p}_* \tilde{q}^* \mathcal{O}_{\mathbb{P}^5}(d) \simeq \text{Sym}^d(\mathcal{G}^\vee)$. Let $s \in H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3))$ be the defining section of Y . Then Altman and Kleiman show that the zero set F' of the section $t \in H^0(\text{Gr}, \text{Sym}^3(\mathcal{G}^\vee))$ which corresponds to $\tilde{q}^*(s)$ represents the functor

$$(\mathbb{C} - \text{schemes}) \rightarrow (\text{Sets}), \quad T \mapsto \{\text{flat families of lines in } Y \subset \mathbb{P}^5 \text{ par. by } T\}.$$

Hence $F' \simeq F$. Next, they show that t is a regular section. We could deduce this also be from [BD], because $\text{rk Sym}^3 \mathcal{G}^\vee = 4 = \text{codim}(F, \text{Gr})$. \square

Corollary 3.3.2.3. *The normal bundle of F in Gr is $\text{Sym}^3(\mathcal{F}^\vee)$.* \square

Lemma 3.3.2.4. *Let \mathcal{E} be a rank 2 vector bundle on a smooth complex variety X . Then $\text{ch}_1(\text{Sym}^3 \mathcal{E}) = 6 \text{ch}_1(\mathcal{E})$ and $\text{ch}_2(\text{Sym}^3 \mathcal{E}) = 2 \text{ch}_1^2(\mathcal{E}) + 10 \text{ch}_2(\mathcal{E})$.* \square

Next, we look at the normal bundle sequence of $F \subset \text{Gr}$

$$0 \rightarrow \mathcal{T}_F \rightarrow \mathcal{T}_{\text{Gr}|F} \rightarrow \mathcal{N}_{F/\text{Gr}} \rightarrow 0. \quad (3.29)$$

We have $\mathcal{T}_{\text{Gr}} = \mathcal{H}om(\mathcal{G}, \mathcal{Q}) \simeq \mathcal{G}^\vee \otimes \mathcal{Q}$ and $\mathcal{N}_{F/\text{Gr}} = \text{Sym}^3(\mathcal{G}^\vee)|_F = \text{Sym}^3(\mathcal{F}^\vee)$. To calculate the Chern classes of these bundles we have to deal with Schubert cocycles. Fix a flag

$$\{0\} = V_0 \subset V_1 \subset \dots \subset V_6 = \mathbb{C}^6. \quad (3.30)$$

Let $4 \geq a_1 \geq a_2 \geq 0$ be two integers. We define the Schubert cycle

$$\Sigma_{a_1, a_2} := \{[l] \in \text{Gr} \mid \dim(W_{[l]} \cap V_{4+i-a_i}) \geq i\}$$

and the corresponding cocycle

$$\sigma_{a_1, a_2} := [\Sigma_{a_1, a_2}] \in H^{2(a_1+a_2)}(\text{Gr}, \mathbb{Z}).$$

If $a_2 = 0$, we will write Σ_{a_1} instead of $\Sigma_{a_1, 0}$ and σ_{a_1} instead of $\sigma_{a_1, 0}$. The Schubert cocycles generate freely the integral cohomology of Gr and are independent of the flag. The Gauss–Bonnet theorem gives

$$c_r(\mathcal{G}) = (-1)^r \sigma_{1, \dots, 1}.$$

We thus get

$$\text{ch}_1(\mathcal{G}) = -\sigma_1$$

and

$$\text{ch}_2(\mathcal{G}) = \frac{\sigma_1^2}{2} - \sigma_{1,1} = \frac{\sigma_{2,0} - \sigma_{1,1}}{2}$$

where we used a special case of Pieri's formula which gives $\sigma_1^2 = \sigma_{1,1} + \sigma_{2,0}$ (see [GH]).

Denote for any compact manifold X by D_X the involution of $\bigoplus_i H^{2i}(X)$ acting as $(-1)^i \text{id}$ on $H^{2i}(X)$. Using $\text{ch}(\mathcal{G}^\vee) = D_X(\text{ch}(\mathcal{G}))$ and $\text{ch}_i(\mathcal{Q}) = -\text{ch}_i(\mathcal{G})$ for $i \geq 1$ we see that

$$\text{ch}_1(\mathcal{T}_{\text{Gr}}) = -4 \text{ch}_1(\mathcal{G}) - 2 \text{ch}_1(\mathcal{G}) = 6\sigma_1$$

and

$$\begin{aligned} \text{ch}_2(\mathcal{T}_{\text{Gr}}) &= 4 \text{ch}_2(\mathcal{G}^\vee) + 2 \text{ch}_2(\mathcal{Q}) + \text{ch}_1(\mathcal{G}^\vee) \text{ch}_1(\mathcal{Q}) \\ &= 4 \text{ch}_2(\mathcal{G}) - 2 \text{ch}_2(\mathcal{G}) + \text{ch}_1^2(\mathcal{G}) \\ &= \sigma_2 - \sigma_{1,1} + \sigma_1^2 \\ &= 2\sigma_2. \end{aligned}$$

For the normal bundle we get using Lemma 3.3.2.4

$$\text{ch}_1(\text{Sym}^3(\mathcal{G}^\vee)) = 6\sigma_1$$

and

$$\begin{aligned} \text{ch}_2(\text{Sym}^3(\mathcal{G}^\vee)) &= 2\sigma_1^2 + 5(\sigma_2 - \sigma_{1,1}) \\ &= 7\sigma_2 - 3\sigma_{1,1}. \end{aligned}$$

We will denote by τ_{a_1, a_2} the pullback of σ_{a_1, a_2} to F , and as above, we write τ_{a_1} for $\tau_{a_1, 0}$. Taking into account (3.29) we have proved the

Proposition 3.3.2.5. *The Chern character of \mathcal{T}_F is*

$$4 + 3\tau_{1,1} - 5\tau_2 + \chi(F) \text{ vol}$$

where vol is the integral generator of $H^8(F, \mathbb{Z})$ given by the orientation. In other words, $c_2(F) = 5\tau_2 - 3\tau_{1,1}$.

3.3.3 The image of the correspondence $[Z]_*$

To begin with, let $C \subset Y$ be a complete intersection of Y with three hyperplanes, let $S_C := \{[l] \in F \mid l \cap C \neq \emptyset\}$.

Proposition 3.3.3.1. *The set S_C is a surface in F with fundamental class*

$$[S_C] = \tau_2 = \frac{1}{8}(c_2(F) + 3g^2).$$

Proof. Assume $C = H_1 \cap H_2 \cap H_3 \cap Y$ where the $H_i \subset \mathbb{P}^5$ are hyperplanes. Let $U \subset \mathbb{C}^6$ denote the three-dimensional subspace corresponding to $H_1 \cap H_2 \cap H_3$. Choose a flag as in (3.30) such that $V_3 = U$. Then

$$S_C = \{[l] \in F \mid \dim W_{[l]} \cap U \geq 1\} = \Sigma_2 \cap F,$$

and therefore $[S_C] = \tau_2$. Since the Plücker embedding is given by the line bundle $\det \mathcal{F}^\vee$, we have $g = c_1(\mathcal{F}^\vee) = \tau_1$. Hence, $g^2 = \tau_1^2 = \tau_{1,1} + \tau_2$ and the claim follows from Proposition 3.3.2.5. \square

Corollary 3.3.3.2. *Let $B(Y) \subset H^*(Y, \mathbb{Q})$ be the \mathbb{Q} -vector space of algebraic cohomology classes, let $A \subset H^*(F, \mathbb{Q})$ be the image of $B(Y)$ under the correspondence $[Z]_* : H^*(Y) \rightarrow H^*(F)$. Then A is generated as a \mathbb{Q} -vector space by $1_F, \text{NS}(F), c_2(F), g^2$.*

Proof. Beauville and Donagi show that the restriction in degree 4 yields an isomorphism of Hodge structures $[Z]_* : H^4(Y, \mathbb{Z}) \simeq H^2(F, \mathbb{Z})$. Combined with the fact that $H^6(Y, \mathbb{Q}) = \mathbb{Q}[C]$ and that the class of the surface S_C considered above in $H^4(F, \mathbb{Q})$ is nothing but $q_* p^*([C]) = [Z]_*([C])$ for a generic choice of the hyperplanes defining C , the assertion follows from Proposition 3.3.3.1. \square

3.3.4 The Fano surface of lines on a cubic threefold

Let $W \subset Y$ be a smooth cubic threefold, i.e. a hyperplane section. Then by a classical theorem of Fano [Fa], the variety F_W parameterizing lines on W is a smooth surface. This is naturally contained in F .

Proposition 3.3.4.1. *The fundamental class of the Fano surface is*

$$[F_W] = c_2(\mathcal{F}) = \frac{1}{8}(5g^2 - c_2(F)).$$

Proof. The Fano surface is the zero set of the section of $p_*\mathcal{O}_Z(1) = \mathcal{F}^\vee$ corresponding to the defining section of W in $H^0(Y, \mathcal{O}_Y(1))$. This section is regular since H is of codimension $2 = \text{rk}(\mathcal{F}^\vee)$. But then $[F_W] = c_2(\mathcal{F}^\vee) = c_2(\mathcal{F})$. For the second equality we use that $c_2(\mathcal{F}) = \tau_{1,1}$ and that $g^2 = \tau_{1,1} + \tau_2$ and we combine this with Proposition 3.3.2.5. \square

Voisin [Vo1] pointed out that F_W is a Lagrangian surface with respect to the holomorphic symplectic form on F . One can see this by noting that the image of the restriction map $H^2(F, \mathbb{Q}) \rightarrow H^2(F_W, \mathbb{Q})$ is identified via the universal line to the image of the restriction $H^4(Y, \mathbb{Q}) \rightarrow H^4(W, \mathbb{Q})$.

Let $\text{NS}(F)$ be the rational Néron–Severi group and let $T(F)$ be its Beauville–Bogomolov orthogonal complement. Identifying $B(H^4(F, \mathbb{Q})) \simeq \text{End}_{\text{Hdg}}(T(F)) \oplus \text{End}_{\mathbb{Q}}(\text{NS}(F))$ (see Section 3.4), we see that F_W is a Lagrangian surface whose fundamental classes has a non-trivial component in $\text{Sym}^2(T(F))$. Indeed, $c_2(F)$ has such a component and $g^2 \in \text{End}_{\mathbb{Q}}(\text{NS}(F))$.

3.4 Discussion

We note that all algebraic classes which we have constructed in the last two sections are linear combinations of products of divisor classes and the second Chern class of the variety. This has the following conceptual explanation.

Let X be a projective deformation of the second Hilbert square of a K3 surface. Recall that Beauville and Bogomolov found a natural, deformation-invariant quadratic form q on $H^2(X, \mathbb{Z})$ such that the rational singular cohomology of X becomes a Hodge structure of K3 type. Denote the corresponding period domain by Ω . Proposition 3.1.1.1 and Lemma 3.1.2.1 allow us to make the identification $H^4(X, \mathbb{Q}) \simeq \text{End}_{\mathbb{Q}}(H^2(X, \mathbb{Q}))'$ where $'$ is the involution induced by adjunction with respect to q . Let $T(X)$ and $\text{NS}(X)$ be the transcendental lattice resp. the Néron–Severi group of $H^2(X, \mathbb{Q})$ (see (1.5)). Then we get by (1.6)

$$B(H^4(X, \mathbb{Q})) \simeq \text{End}_{\text{Hdg}}(H^2(X, \mathbb{Q}))' \simeq \text{End}_{\text{Hdg}}(T(X))' \oplus \text{End}_{\mathbb{Q}}(\text{NS}(X))'.$$

It is easy to see that $c_2(X)$ is a nonzero multiple of the identity (see [Vo4, p. 18f.] or [NW]).

For $\alpha_1, \dots, \alpha_r \in \text{NS}(X)$, let $\Omega_{\alpha_i} \subset \Omega_{\alpha}$ be the corresponding Hodge loci (see Definition 1.2.1.1). We write $\Omega_{\alpha_1, \dots, \alpha_r}$ for $\Omega_{\alpha_1} \cap \dots \cap \Omega_{\alpha_r}$.

Definition 3.4.1. Let $\varphi \in B(H^4(X, \mathbb{Q})) \subset \text{End}_{\mathbb{Q}}(H^2(X, \mathbb{Q}))$. We say that the Hodge locus $\Omega_{\varphi} \subset \Omega$ of φ contains a divisor-like deformation space if there exist $\alpha_1, \dots, \alpha_r \in \text{NS}(X) \setminus \{0\}$ with

$$\Omega_{\alpha_1, \dots, \alpha_r} \subset \Omega_{\varphi}.$$

Proposition 3.4.2. Let $\varphi \in B(H^4(X, \mathbb{Q}))$. If the Hodge locus of φ contains a divisor-like deformation space, then φ is a linear combination of $c_2(X)$ and of products of divisor classes.

Proof. If φ is not a linear combination of $c_2(X)$ and products of divisor classes, then $\varphi(\sigma_X) = \lambda\sigma_X$ for some $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. We show that this implies that Ω_{φ} does not contain a divisor-like deformation space.

Let $W_{\lambda} \subset H^2(X, \mathbb{C})$ be the eigenspace corresponding to λ , then by Lemma 1.2.1.2 we have $\Omega_{\varphi} = \mathbb{P}(W_{\lambda}) \cap \Omega$. Note that W_{λ} contains no nonzero rational vector, since $\lambda \notin \mathbb{Q}$.

On the other hand, let $\alpha_1, \dots, \alpha_r \in \text{NS}(X) \setminus \{0\}$, put

$$W_{\alpha_1, \dots, \alpha_r} := \ker(q(\alpha_1, *)) \cap \dots \cap \ker(q(\alpha_r, *)).$$

Then $\Omega_{\alpha_1, \dots, \alpha_r} = \mathbb{P}(W_{\alpha_1, \dots, \alpha_r}) \cap \Omega$. Note that $W_{\alpha_1, \dots, \alpha_r}$ is defined over \mathbb{Q} .

Clearly $\Omega_{\alpha_1, \dots, \alpha_r} \subset \Omega_{\varphi}$ if and only if $W_{\alpha_1, \dots, \alpha_r} \subset W_{\lambda}$. But $W_{\lambda} \cap W_{\alpha_1, \dots, \alpha_r} = \{0\}$, since W_{λ} does not contain nonzero rational vectors. \square

Now the point is, that the constructions presented in the last sections yield algebraic cohomology classes on X whose Hodge locus contains a divisor-like deformation space.

Indeed, the germ $(\Omega, [\sigma_X])$ is a germ of a universal deformation space of X . In the case that $X = \text{Hilb}^2(S)$ for a K3 surface S and that $\mathcal{L} \in \text{Pic}(S)$ the vector bundle $\mathcal{L}^{[2]}$ deforms (at least, there could be other deformation directions) in all directions in which the classes $\varphi(c_1(\mathcal{L}))$ and δ remain of type $(1,1)$. Thus the Hodge loci of the characteristic classes $c_2(\mathcal{L}^{[2]})$ and $\text{ch}_2(\mathcal{L}^{[2]})$ contain divisor-like deformation spaces. The proposition shows that these classes are linear combinations of $c_2(X)$ and of divisor classes.

The case of the Fano surface and the surface of lines meeting a given curve in $X = F(Y)$ is analogous, these surfaces exist whenever the Plücker polarization remains algebraic.

Similar observations apply to many examples of surfaces one could construct on X . For example, a Lagrangian surface $S \subset X$ has a fundamental class with Hodge locus containing a divisor-like deformation space by a result of Voisin [Vo1].

In the beginning of this chapter we started with a K3 surface S . The idea was to construct algebraic cycles on a deformation of $\text{Hilb}^2(S)$ and to deform such cycles to $\text{Hilb}^2(S)$ in order to prove the algebraicity of Hodge classes on $S \times S$. Clearly, the Hodge classes on $S \times S$ we are interested in correspond to Hodge classes in $B(H^4(\text{Hilb}^2(S), \mathbb{Q}))$ which lie in $\text{End}_{\text{Hdg}}(T(\text{Hilb}^2(S)) \setminus \mathbb{Q})$. The proposition tells us that in order to construct algebraic cycles which represent such classes, we have to find a construction which leads to cohomology classes without divisor-like deformation space in their Hodge locus.

Chapter 4

Two complementary results

4.1 K3 surfaces with CM are defined over number fields

In this section we study K3 surfaces with complex multiplication. The following result was known before by work of Piatetski-Shapiro and Shafarevich [PSS], later it was reproved by Rizov [Ri] (see (iii) in the remark below). However, the method presented here is different.

Theorem 4. *Let S be a K3 surface with complex multiplication by a CM field E . Assume that $m = \dim_E T(S) = 1$. Then S is defined over an algebraic number field.*

Proof. Let $\varphi \in E = \text{End}_{\text{Hdg}}(T(S))$ with $\varphi(\sigma_S) = \lambda\sigma_S$ such that $E = \mathbb{Q}(\lambda)$. Here we write σ_S for the period of S . Denote by $T_\lambda \subset T(S)_\mathbb{C}$ the eigenspace to λ . Let Ω be the period domain parametrizing Hodge structures of K3 type on $H^2(S, \mathbb{Z})$ equipped with the quadratic form given by the intersection form on S . By Lemma 1.2.1.2, the Hodge locus of φ in Ω is contained in $\mathbb{P}(T_\lambda) \cap \Omega$. Since $\dim_{\mathbb{C}} T_\lambda = \dim_E(T(S)) = 1$ (see the proof of Corollary 1.1.8.2 for the argument), we see that the Hodge locus of φ is reduced to the point $[\sigma_S]$.

The idea is now to consider a moduli space defined over $\overline{\mathbb{Q}}$ which contains (S, φ) and to use the rigidity of (S, φ) to conclude that S is defined over $\overline{\mathbb{Q}}$.

Mukai's Theorem 1.1.4.1 tells us that there are surfaces $Z_1, \dots, Z_k \subset S \times S$ and rational numbers a_1, \dots, a_k such that $\varphi = \sum_i a_i [Z_i]_{S \times S}$. Embed S in \mathbb{P}^N and denote by $\text{Hilb}^{[S]}(\mathbb{P}^N)$ (resp. by $\text{Hilb}^{[Z_i]}(\mathbb{P}^N \times \mathbb{P}^N)$) the Hilbert scheme of subvarieties of \mathbb{P}^N with the same Hilbert polynomial as S (resp. subvarieties of $\mathbb{P}^N \times \mathbb{P}^N$ with the same Hilbert polynomial as Z_i). Define \tilde{H} as the incidence variety

$$\tilde{H} = \left\{ (T, Y_1, \dots, Y_k) \in \text{Hilb}^{[S]}(\mathbb{P}^N) \times \prod_i \text{Hilb}^{[Z_i]}(\mathbb{P}^N \times \mathbb{P}^N) \mid Y_i \subset T \times T \right\}$$

Then \tilde{H} is a projective \mathbb{Q} -variety which admits $x_0 = (S, Z_1, \dots, Z_k)$ as a \mathbb{C} -rational point. Choose an irreducible component H of $\tilde{H}_{\overline{\mathbb{Q}}}$ which is defined over $\overline{\mathbb{Q}}$ and which passes through x_0 . Denote by $\rho : \mathcal{Y} \rightarrow H$ the universal family over H . Let $\pi : \mathcal{X} \rightarrow H$ be the pullback of the universal family over $\text{Hilb}^{[S]}(\mathbb{P}^N)$ along the composition

$$H \rightarrow \text{Hilb}^{[S]}(\mathbb{P}^N) \times \prod_i \text{Hilb}^{[Z_i]}(\mathbb{P}^N \times \mathbb{P}^N) \rightarrow \text{Hilb}^{[S]}(\mathbb{P}^N).$$

The fibers of ρ are products of surfaces $T \times Y_1 \times \dots \times Y_k$ with $Y_i \subset T \times T$ and the fibers of π are surfaces T . There is a commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\quad} & \mathcal{X} \\ & \searrow \rho & \swarrow \pi \\ & & H \end{array} \quad (4.1)$$

where the horizontal map is given by the restriction of the projection $H \times \mathbb{P}^N \times \prod_i (\mathbb{P}^N \times \mathbb{P}^N) \rightarrow H \times \mathbb{P}^N$ to \mathcal{Y} . Since $\pi^{-1}(x_0) \simeq S$ is smooth and since π is flat there exists an open subset $U \subset H$ such that $\pi : \pi^{-1}(U) \rightarrow U$ is smooth (see [Har], Exercise III.10.2, we may assume without loss of generality H reduced).

Now we show, that all fibers of $\pi_{\mathbb{C}} : \pi^{-1}(U)_{\mathbb{C}} \rightarrow U_{\mathbb{C}}$ are isomorphic to S . For that sake we first pass to the analytic category (without changing the notation). Clearly all fibers of π over $U_{\mathbb{C}}$ are smooth K3 surfaces. For $x \in U_{\mathbb{C}}$, choose a path $\gamma = (x_t)_{t \in [0,1]}$ in $U_{\mathbb{C}}$ connecting x_0 and $x = x_1$. Choosing a marking of the restriction of \mathcal{Y} to γ , we get a path $[\sigma_{x_t}]$ in the period domain Ω . All Hodge structures along this path are contained in the Hodge locus of φ . Indeed, the self-products of all surfaces parametrized by γ contain surfaces of class $[Z_i]$ as can be read off the diagram (4.1). Hence φ is an algebraic class on the self-products of the corresponding K3 surfaces. By the rigidity of (S, φ) we see that $[\sigma_t] = [\sigma_0]$ for all t . The local Torelli theorem for K3 surfaces implies that $\mathcal{X}_x \simeq \mathcal{X}_{x_0} = S$.

It remains to note that the $\overline{\mathbb{Q}}$ -rational points are dense in H and so there is a $\overline{\mathbb{Q}}$ -rational point p contained in U . The fiber $\pi^{-1}(p) \simeq S$ is defined over $\overline{\mathbb{Q}}$. \square

Remark. (i) In this proof, we used Mukai's result for Picard number smaller than 5. Indeed, it could happen for a K3 surface S as in the theorem that the Picard number $\rho(S)$ is smaller than 5. This would mean that $[E : \mathbb{Q}] = 16$ or $[E : \mathbb{Q}] = 18$. There exist alternative proofs in these two cases avoiding the reference to Mukai, see (iii).

(ii) The proof actually shows that the surfaces $Z_1, \dots, Z_k \subset S \times S$ with $\sum_i a_i [Z_i] = \varphi$ can be chosen in such a way that they are defined over $\overline{\mathbb{Q}}$.

(iii) Some authors use a stricter notion of a K3 surface with complex multiplication. They say that S has CM if and only if $\text{SMT}(T(S))$ is a torus (see e.g. [PSS], [Bo], [Ri]). We will see below that this is equivalent to the assumption of Theorem 4.

Piatetski-Shapiro and Shafarevich [PSS] prove that a K3 surface with complex multiplication in this stricter sense is defined over an algebraic number field. Their proof relies on the Kuga–Satake correspondence and on the corresponding result for Abelian varieties. Rizov [Ri] proves that the period map for K3 surfaces is defined over \mathbb{Q} . From this he deduces that if a K3 surface has complex multiplication by E in the strict sense, then S is defined over a cyclic, finite extension of E .

Now we prove that complex multiplication in the strict sense coincides with the assumption of Theorem 4. Let S be a K3 surface with $\text{End}_{\text{Hdg}}(T(S)) = E$ for a CM field E . Then Zarhin’s results [Z] imply that the special Mumford–Tate group of $T(S)$ is the Weil restriction of the unitary group of a certain non-degenerate Hermitian form on the E -vector space $T(S)$.

Combining this with Corollary 1.1.8.2, we see that a K3 surface S has complex multiplication by a CM field E with $\dim_E(T(S)) = 1$ if and only if the special Mumford–Tate group of $T(S)$ is a torus. Indeed, the unitary group of a non-degenerate Hermitian form in m variables is a torus if and only if $m = 1$ and the special orthogonal group of a non-degenerate quadratic form in $m \geq 3$ variables is never a torus.

(iv) Shioda and Inose [SI] show that a K3 surface S with Picard number 20 is defined over an algebraic number field. Such a K3 surface has complex multiplication by a quadratic CM field E and $\dim_E T(S) = 1$ (see (iii)). Their proof relies on a structure theorem for K3 surfaces with maximal Picard number. This says that a K3 surface with Picard number 20 is a quotient of a Kummer surface associated to the self-product of an elliptic curve with complex multiplication.

4.2 André motives

In this part we are concerned with André's category of motives which will be reviewed in the first two sections. The basic idea is to modify Grothendieck's homological motives by adjoining formally the inverse Lefschetz operator to the morphism sets. This allows to circumvent the open standard conjectures and to get a semisimple, Tannakian category of motives.

We then focus on algebraic deformations of moduli spaces of sheaves on K3 surfaces. These form one of the few classes of examples of irreducible symplectic varieties. The geometry of these has been studied extensively by many people, among them Beauville, Huybrechts, O'Grady and Verbitsky. Many results suggest that much of the geometry is governed by the weight two Hodge structure on the second integral cohomology which can be equipped with Beauville's quadratic form. Our main result in this section is another manifestation of this principle. We show that for any algebraic deformation Y of a moduli space of sheaves on a K3 surface, the motive $\mathfrak{h}(Y)$ is an object of the category generated by $\mathfrak{h}^2(Y)$. This result is actually a reformulation of Markman's result on the monodromy group of moduli spaces of sheaves on K3 surfaces in motivic terms.

Using André's results on motives of varieties with a polarized Hodge structure of primitive K3 type, we can conclude that any Hodge class on Y is motivated and therefore absolute in the sense of Deligne. Furthermore, the proof of the standard conjectures for all smooth projective varieties would imply the Hodge conjecture for Y .

4.2.1 Tensor categories and Tannakian categories

The definitions given in this section are taken from [An3, Ch. 2]. Let k be a field, let \mathcal{T} be a k -linear category (i.e. \mathcal{T} is an additive category, the morphism sets in \mathcal{T} are k -vector spaces and the composition of morphisms is k -bilinear).

Definition 4.2.1.1. The category \mathcal{T} is a *tensor category* if it admits a tensor structure that is

- a bilinear bifunctor

$$\otimes : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T},$$

- a unit object $\mathbb{1}$ and
- functorial isomorphisms

$$\begin{aligned} a_{LMN} &: (L \otimes M) \otimes N \xrightarrow{\sim} L \otimes (M \otimes N) \\ c_{MN} &: M \otimes N \xrightarrow{\sim} N \otimes M \\ u_M &: M \otimes \mathbb{1} \xrightarrow{\sim} M \text{ and } u'_M : \mathbb{1} \otimes M \xrightarrow{\sim} M \end{aligned}$$

which are subject to the natural compatibility conditions.

A tensor category is called *rigid* if it admits in addition an autoduality

$$\vee : \mathcal{T}^{\text{op}} \rightarrow \mathcal{T}$$

such that for any object M of \mathcal{T} the pairs of functors $(* \otimes M^\vee, * \otimes M)$ and $(M^\vee \otimes *, M \otimes *)$ are adjoint pairs.

Examples. i) The category Vec_k of finite-dimensional k -vector spaces with the classical tensor product and duality is a rigid tensor category.

ii) The category of \mathbb{Z} -graded, finite-dimensional k -vector spaces VecGr_k has a tensor structure. The tensor product is given by the graded tensor product

$$\left(\bigoplus_k V_k\right) \otimes \left(\bigoplus_l W_l\right) = \bigoplus_m \bigoplus_{k+l=m} (V_k \otimes_k W_l),$$

the isomorphisms a, c, u defined as the natural ones.

Suppose that the rigid, k -linear tensor category \mathcal{T} is Abelian and that $\text{End}(\mathbb{1}) = k$.

Definition 4.2.1.2. A *fibre functor* on \mathcal{T} is a faithful, exact functor of tensor categories (i.e. compatible with the tensor structures)

$$\omega : \mathcal{T} \rightarrow \text{Vec}_K$$

where K is some field extension of k .

If \mathcal{T} admits a fibre functor, it is called *Tannakian*. If \mathcal{T} admits a fibre functor to Vec_k , it is called *neutral Tannakian*.

Definition 4.2.1.3. A *Tannakian subcategory* of a Tannakian category \mathcal{T} is a full subcategory $\mathcal{T}' \subset \mathcal{T}$ which is stable under finite sums, tensor products and duals and such that each subobject or quotient in \mathcal{T} of an object in \mathcal{T}' is in \mathcal{T}' .

4.2.2 André motives

Let X be a smooth projective variety over \mathbb{C} , polarized by an ample divisor H . Denote by L the Lefschetz operator on the cohomology ring associated with H , given by the cup product with the fundamental class $[H]$. The hard Lefschetz theorem asserts that for $k \leq d = \dim(X)$

$$L^k : H^{d-k}(X, \mathbb{Q}) \rightarrow H^{d+k}(X, \mathbb{Q}) \quad (4.2)$$

is an isomorphism.

We define the *Lefschetz involution* $*_L$ as the operator on $H^*(X, \mathbb{Q})$ which maps $H^{d-k}(X, \mathbb{Q})$ to $H^{d+k}(X, \mathbb{Q})$ via the Lefschetz isomorphism and which maps

$H^{d+k}(X, \mathbb{Q})$ to $H^{d-k}(X, \mathbb{Q})$ via the inverse of the Lefschetz isomorphism for all $k \leq d$. In formulas

$$*_L(\alpha) = L^{d-k}(\alpha)$$

for all $0 \leq k \leq 2d$ and for all $\alpha \in H^k(X, \mathbb{Q})$.

Note that $*_L$ respects the Hodge decomposition and therefore it can be interpreted as a Hodge class on $X \times X$. There is the famous

Grothendieck's standard conjecture $B(X)$. *The operator $*_L$ is given by an algebraic class.*

The conjecture is true for curves and surfaces by the Lefschetz theorem on $(1, 1)$ -classes. Lieberman could verify it for Abelian varieties (cf. [Kl]). Arapura [Ar] showed that it holds for moduli spaces of sheaves on curves, Abelian or K3 surfaces. The conjecture is easily seen to be stable under products.

Definition 4.2.2.1 (André, cf. [An1], 2.1). A cohomology class $\alpha \in H^*(X, \mathbb{Q})$ is *motivated* if there exist a smooth projective variety Y and algebraic cycles Z_1, Z_2 on $X \times Y$ with

$$\alpha = (\text{pr}_X^{X \times Y})_*([Z_1] \cup *_XY[Z_2]).$$

Here $*_{XY}$ is the Lefschetz involution associated to some product polarization $(\text{pr}_X^{X \times Y})^*H + (\text{pr}_Y^{X \times Y})^*G$ where H (resp. G) is an ample divisor on X (resp. on Y).

Denote the set of motivated classes on X by $A_{\text{mot}}(X)$.

Some properties of motivated classes (see [An1, 2.1 - 2.5]).

i) Any algebraic cohomology class $[Z]$ is motivated (take $Y = \text{Spec}(\mathbb{C})$, $Z_1 = (\text{pr}_X^{X \times Y})^*Z$ and $Z_2 = ((\text{pr}_X^{X \times Y})^*H^{\dim X}) \times Y$).

Vice versa, if $B(Y)$ was true for all smooth, projective varieties Y , then any motivated class would be algebraic.

ii) The set of motivated cohomology classes is stable under sums and cup product.

iii) For any smooth, projective X, Z

$$\begin{aligned} (\text{pr}_X^{X \times Z})^*(A_{\text{mot}}(X)) &\subset A_{\text{mot}}(X \times Z) \text{ and} \\ (\text{pr}_X^{X \times Z})_*(A_{\text{mot}}(X \times Z)) &\subset A_{\text{mot}}(X). \end{aligned}$$

iv) The class of $*_L : H^*(X, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$ as well as the classes of the Künneth projectors $\pi_{H^i} : H^*(X, \mathbb{Q}) \rightarrow H^i(X, \mathbb{Q})$ are motivated on $X \times X$.

v) One could work in the definition of $A_{\text{mot}}(X)$ with another Weil cohomology which is comparable to the singular cohomology to get a similar definition of $A_{\text{mot}}(X)$ (see [An1, 2.3] for a precise statement).

vi) Any motivated Hodge class is absolute in the sense of Deligne (cf. [De], 2.10 or [Vo3], 3.1).

Item ii) shows that $A_{\text{mot}}(X)$ is an Abelian group. Items ii), iii) and iv) can be used to show that the grading on $H^*(X, \mathbb{Q})$ induces a grading on $A_{\text{mot}}^\bullet(X)$.

Definition 4.2.2.2. Let $X = \bigsqcup_i X_i$ and $Y = \bigsqcup_j Y_j$ be finite disjoint unions of smooth, projective, connected varieties and let $r : X \times Y \rightarrow \mathbb{Z}$ be a locally constant function, i.e. a collection of integers $r_{i,j}$ for each $X_i \times Y_j$.

A *motivated correspondence from X to Y of degree r* is an element of $\bigoplus_{\text{mot}}^{\dim X_i + r_{i,j}}(X_i \times Y_j)$.

The graded space of motivated correspondences from X to Y is denoted by $C_{\text{mot}}^\bullet(X, Y)$.

Properties ii) and iii) above allow one to define the composition of $f \in C_{\text{mot}}^r(X, Y)$ and $g \in C_{\text{mot}}^s(Y, Z)$ by the rule

$$g \circ f = (\text{pr}_{X \times Z}^{X \times Y \times Z})_* ((\text{pr}_{X \times Y}^{X \times Y \times Z})^* f \cup (\text{pr}_{Y \times Z}^{X \times Y \times Z})^* g) \in C_{\text{mot}}^{r+s}(X, Z).$$

Here $(r+s)_{i,l} = \sum_j r_{i,j} + s_{j,l}$.

In particular, $C_{\text{mot}}^\bullet(X, X)$ carries the structure of a graded algebra.

Given a morphism $f : X \rightarrow Y$ we define

$$f^* := [{}^t\Gamma_f] \in C_{\text{mot}}^0(Y, X) \text{ and } f_* := [\Gamma_f] \in C_{\text{mot}}^{\dim X - \dim Y}(X, Y) \quad (4.3)$$

where $\Gamma_f \subset X \times Y$ is the graph of f and ${}^t\Gamma_f$ is the image of Γ_f under the natural isomorphism $X \times Y \simeq Y \times X$.

Recall that an element a of a ring is an *idempotent* if $a^2 = a$.

Definition 4.2.2.3 (André, cf. [An1], 4.2). The *category of André motives* \mathcal{M} is given by the

objects: triples (X, n, q) where X is a smooth, projective variety over \mathbb{C} , n is a \mathbb{Z} -valued locally constant function on X and $q \in C_{\text{mot}}^0(X)$ is an idempotent. We write $q\mathfrak{h}X(n)$ for (X, n, q) .

morphisms: $\text{Hom}_{\mathcal{M}}(q\mathfrak{h}(X)(r), q'\mathfrak{h}(Y)(r')) = q' \circ C_{\text{mot}}^{r'-r}(X, Y) \circ q$.

The motive $(X, 0, \pi_{H^i})$ is denoted by $\mathfrak{h}^i(X)$ (see iv) above). Note that the motive $\mathfrak{h}(X) := (X, 0, \text{id})$ decomposes as a direct sum $\mathfrak{h}(X) = \bigoplus_i \mathfrak{h}^i(X)$.

The motive $\mathfrak{h}(\text{Spec}(\mathbb{C}))$ is denoted by $\mathbb{1}$, it plays the role of the unit object in \mathcal{M} . The motive $\mathbb{1}(-1) = (\text{Spec}(\mathbb{C}), -1, \text{id})$ is called the *Lefschetz motive*. The motive $\mathbb{1}(1) = (\text{Spec}(\mathbb{C}), 1, \text{id})$ (which is the dual of the Lefschetz motive, see below) is called the *Tate motive*.

There is a functor

$$\mathfrak{h} : \begin{cases} P(\mathbb{C})^{\text{op}} \rightarrow \mathcal{M} \\ X \mapsto \mathfrak{h}(X) \end{cases}$$

where $P(\mathbb{C})$ is the category of smooth, projective \mathbb{C} -varieties. This functor maps a morphism $f : X \rightarrow Y$ to f^* defined in (4.3). The (rational) singular cohomology functor on $P(\mathbb{C})$ factors over \mathfrak{h} , the resulting functor $H^\bullet : \mathcal{M} \rightarrow \text{VecGr}_{\mathbb{Q}}$ is called the Betti realization.

Theorem 4.2.2.4 (André, cf. [An1], 4.3). *The \mathbb{Q} -linear category \mathcal{M} is neutral Tannakian, semisimple and graded. The Betti realization is a conservative, graded fibre functor.*

Recall that a functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is called conservative if it reflects isomorphisms. This means that a morphism f in \mathcal{C} is an isomorphism if and only so is $F(f)$.

Proof. We only give the definition of sums, products and duals.

- For $q\mathfrak{h}(X)(r)$ and $q'\mathfrak{h}(Y)(r')$ in \mathcal{M} the direct sum is defined

$$q\mathfrak{h}(X)(r) \oplus q'\mathfrak{h}(Y)(r') = (q, p)\mathfrak{h}(X \sqcup Y)(r \sqcup r').$$

Here, $r \sqcup r'$ is the function on $X \sqcup Y$ whose restriction to X resp. Y is r resp. r' .

- The tensor structure is given by the product

$$q\mathfrak{h}(X)(r) \otimes q'\mathfrak{h}(Y)(r') = (q \times q')\mathfrak{h}(X \times Y)(r + r').$$

- The unit object is $\mathbb{1} = \mathfrak{h}(\text{Spec}(\mathbb{C}))$.
- The associativity and the unit isomorphisms are given by the natural maps. In order to define an isomorphism

$$c : (q \times q')(\mathfrak{h}(X \times Y))(r + r') \xrightarrow{\sim} (q' \times q)(\mathfrak{h}(Y \times X))(r' + r)$$

it is tempting to define c as $(q' \times q) \circ \sigma^* \circ (q \times q')$ where $\sigma : Y \times X \simeq X \times Y$ is the natural isomorphism. However, with this definition, c would act on the cohomology as $x \otimes y \mapsto (-1)^{\deg x \cdot \deg y} y \otimes x$ where we use the Künneth isomorphism to identify $H^*(X \times Y)$ with $H^*(X) \otimes H^*(Y)$. This means that c would prevent the Betti realization from being a tensor functor. For this reason, André defines c as $(q' \times q) \circ \widetilde{\sigma}^* \circ (q \times q')$ where $\widetilde{\sigma}^*$ is a twisted version of σ^* which makes the sign disappear. In this definition we use that the Künneth projectors are motivated.

- The dual of $q\mathfrak{h}(X)(r)$ is defined as ${}^tq\mathfrak{h}(X)(\dim X - r)$. Here, for a cohomology class $\alpha \in H^*(X \times X)$, the symbol ${}^t\alpha$ denotes the pullback of α along the automorphism of $X \times X$ given by exchanging the factors. \square

We close this subsection by citing André's deformation principle. This gives a positive answer to Grothendieck's invariant cycle conjecture (this conjecture is stated in a precise form in Section 1.2.2) in the motivated world.

Theorem 4.2.2.5 (André, see [An1], 5.1). *Let $f : \mathcal{X} \rightarrow S$ be a smooth projective morphism where S is a smooth, connected, algebraic variety. Let $s \in S$ be a closed point, $m, n \in \mathbb{N}$ and let*

$$\alpha \in H^*(\mathcal{X}_s, \mathbb{Q})^{\otimes n} \otimes (H^*(\mathcal{X}_s, \mathbb{Q})^\vee)^{\otimes m}$$

be a motivated class which is invariant under a subgroup of finite index of $\pi_1(S, s)$ (acting on $H^(S, \mathbb{Q})$ via the monodromy representation). Then any translate of α under parallel transport to $H^*(\mathcal{X}_t, \mathbb{Q})^{\otimes n} \otimes (H^*(\mathcal{X}_t, \mathbb{Q})^\vee)^{\otimes m}$ for $t \in S$ is motivated on \mathcal{X}_t .*

4.2.3 Markman's results

Let S be a projective K3 surface, polarized by an ample divisor H . The Todd genus of S is $1 + 2[x]$ where x is an arbitrary point of S . Its square root is given by $1 + [x]$.

For any coherent sheaf E on S define the Mukai vector by

$$v(E) = \text{ch}(E)\sqrt{\text{td}(S)}.$$

We associate with S a natural weight two Hodge structure

$$\tilde{H}(S, \mathbb{Z}) := H^0(S, \mathbb{Z})(-1) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})(1)$$

and there is also a rational version $\tilde{H}(S, \mathbb{Q})$.

There is a natural duality operator

$$D_S : \tilde{H}(S, \mathbb{Q}) \rightarrow \tilde{H}(S, \mathbb{Q})$$

acting as $(-1)^i \text{id}$ on $H^{2i}(S, \mathbb{Q})$. Since the Künneth components of the diagonal in $S \times S$ are algebraic, D_S is given by an algebraic class.

The Mukai pairing on $\tilde{H}(S, \mathbb{Q})$ is given by

$$\langle \alpha, \beta \rangle = - \int_S D_S(\alpha) \cup \beta.$$

This is a non-degenerate, symmetric bilinear form of signature $(4+, 20-)$.

Let now $v \in H^*(S, \mathbb{Q})$. Assume that there exists a non-empty, compact, fine moduli space $X := M_H(v)$ parameterizing stable sheaves on S with Mukai vector v . Then by results of Mukai, O'Grady and Huybrechts, $M_H(v)$ is a smooth, projective, irreducible symplectic variety of dimension $d = \langle v, v \rangle + 2$ which is deformation equivalent to $\text{Hilb}^{\frac{d}{2}}(S)$. We assume that $d > 2$.

Let \mathcal{E} be a universal sheaf on $S \times X$. Then \mathcal{E} is uniquely determined up to the twist by the pull-back of a line bundle from X . Denote by $p : S \times X \rightarrow S$ and by $q : S \times X \rightarrow X$ the projections. Define

$$\begin{aligned} \varphi'_1 : \tilde{H}(S, \mathbb{Q}) &\rightarrow H^2(X, \mathbb{Q}) \\ \alpha &\mapsto \pi_{H^2} \{ q_* (\text{ch}(\mathcal{E}) \cup p^* \sqrt{\text{td}(S)} \cup p^* D_S(\alpha)) \}. \end{aligned}$$

According to a result of O'Grady [O'G], the restriction of φ'_1 to v^\perp is an isomorphism of Hodge structures (even over \mathbb{Z}).

We normalize the correspondence $\text{ch}(\mathcal{E})p^*\sqrt{\text{td}(S)}$ following [Mar2, Lemma 3.1]: let $\eta := \varphi'_1(v)\langle v, v \rangle^{-1} \in H^2(X, \mathbb{Q})$. Put

$$u := \text{ch}(\mathcal{E}) \cup p^*\sqrt{\text{td}(S)} \cup q^*\exp(\eta). \quad (4.4)$$

Then u is independent of the universal sheaf \mathcal{E} and we define

$$\begin{aligned} \varphi_1 : \tilde{H}(S, \mathbb{Q}) &\rightarrow H^2(X, \mathbb{Q}) \\ \alpha &\mapsto \pi_{H^2}\{q_*(u \cup p^*D_S(\alpha))\}. \end{aligned}$$

Note that

$$\varphi_1(\alpha) = \varphi'_1(\alpha) - \frac{\langle \alpha, v \rangle}{\langle v, v \rangle} \varphi'_1(v).$$

This implies that $\varphi_1(v) = 0$ and that $\varphi_1|_{v^\perp} = \varphi'_1|_{v^\perp}$.

Next, we note that φ_1 is an algebraic correspondence. This is, because u and D_S are so and because the projection $H^*(X, \mathbb{Q}) \rightarrow H^2(X, \mathbb{Q})$ is algebraic (cf. [Ar] where the conjecture B is shown for X).

Since the standard conjecture B holds for S as well, there is an algebraic right inverse $\psi : H^2(X, \mathbb{Q}) \rightarrow \tilde{H}(S, \mathbb{Q})$ (see [Kl, Cor. 3.14]). Since the (Mukai-)orthogonal projection $\tilde{H}(S, \mathbb{Q}) \rightarrow \mathbb{Q}v$ is given on $S \times S$ by the class $-\langle v, v \rangle^{-1}D_S(v) \otimes v$, the orthogonal projection $\tilde{H}(S, \mathbb{Q}) \rightarrow v^\perp$ is algebraic. Thus we may assume that ψ induces an isomorphism

$$\psi : H^2(X, \mathbb{Q}) \xrightarrow{\sim} v^\perp \subset \tilde{H}(S, \mathbb{Q}) \quad (4.5)$$

which is inverse to $\varphi_1|_{v^\perp}$.

Let G_v be the fix group of v in $\text{Aut}(\tilde{H}(S, \mathbb{Z}), \langle \cdot, \cdot \rangle)$. Markman defines two representations of G_v on $H^*(X, \mathbb{Z})$. We will now describe both of them.

1.) Let p_{ij} be the projection from $X \times S \times X$ to the (i, j) -th factor. For $g \in G_v$ set

$$\gamma'_g := (p_{13})_* \left(p_{12}^* D_{X \times S} ((\text{id} \otimes g)({}^t u)) \cup p_{23}^* u \right)^{-1} \in H^*(X \times X, \mathbb{Q}),$$

where $D_{X \times S}$ is the duality operator acting by $(-1)^i$ on $H^{2i}(X \times S, \mathbb{Q})$ and the class u was introduced in (4.4). Let $l : H^*(X, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$ be the universal polynomial map which takes the Chern character $(r + a_1 + a_2 + \dots)$ of a coherent sheaf to its total Chern class $(1 + a_1 + (\frac{a_1^2}{2} - a_2) + \dots)$. Then by definition

$$\gamma_g := \text{degree } d \text{ part of } l(\gamma'_g) = c_d(\gamma'_g).$$

Theorem 4.2.3.1 (Markman, cf. [Mar2], Thm. 3.10 and Cor. 3.14). *i) For $g \in G_v$ the correspondence γ_g acts as a (degree-preserving) automorphism on $H^*(X, \mathbb{Q})$.*

ii) The map

$$\begin{aligned} \gamma : G_v &\rightarrow \text{Aut}(H^*(X, \mathbb{Q})) \\ g &\mapsto \gamma_g \end{aligned}$$

is a faithful representation of G_v .

iii) The class $u \in \tilde{H}(S, \mathbb{Q}) \otimes H^(X, \mathbb{Q})$ is invariant under the product representation of G_v , where G_v acts on the left hand factor via the natural representation.*

The theorem implies that the algebraic maps

$$\begin{aligned} \varphi_i : \tilde{H}(S, \mathbb{Q})(-i) &\rightarrow H^{2i}(X, \mathbb{Q}) \\ \alpha &\mapsto \pi_{H^{2i}}\{q_*(u \cup p^*D_S(\alpha))\} \end{aligned} \tag{4.6}$$

are G_v -equivariant.

2.) To define the second representation of G_v note that the space $\tilde{H}(S, \mathbb{Q})$ has four positive directions. Given any two positive four-spaces F and F' , orientations of these spaces can be compared using orthogonal projections. An isometry $g \in \text{Aut}(\tilde{H}(S, \mathbb{Q}), \langle \cdot, \cdot \rangle)$ is called *orientation preserving* if for an oriented, positive four-space F the space $g(F)$ has the same orientation. This induces the *covariance or orientation character*

$$\text{cov} : G \rightarrow \mathbb{Z}/2\mathbb{Z} \tag{4.7}$$

sending g to 0 or 1 according to whether it preserves orientations or not.

Then Markman defines the representation

$$\begin{aligned} \gamma_{\text{mon}} : G_v &\rightarrow \text{Aut}(H^*(X, \mathbb{Q})) \\ g &\mapsto (D_X)^{\text{cov}(g)} \circ \gamma_g \end{aligned}$$

where again D_X is the duality operator of X , acting by $(-1)^i \text{id}$ on $H^{2i}(X)$.

The subscript is justified by the following result of Markman. An element $g \in \text{Aut}(H^*(X, \mathbb{Q}))$ is called a *monodromy operator* if there exists a deformation $\mathcal{X} \rightarrow B$ with fibre $\mathcal{X}_b = X$ for some $b \in B$ and a $\tilde{g} \in \pi_1(B, b)$ such that g is the image of \tilde{g} under the monodromy representation $\pi_1(B, b) \rightarrow \text{Aut}(H^*(X, \mathbb{Q}))$. Let $\text{Mon}(X)$ be the subgroup of $\text{Aut}(H^*(X, \mathbb{Q}))$ generated by monodromy operators.

Theorem 4.2.3.2 (Markman, cf. [Mar2], Thm. 1.6). *The image of the representation $\gamma_{\text{mon}} : G_v \rightarrow \text{Aut}(H^*(X, \mathbb{Q}))$ is a normal subgroup of finite index in $\text{Mon}(X)$.*

In particular, since γ and γ_{mon} coincide on the kernel N' of the orientation character, its image $N := \gamma(N')$ in $\text{Aut}(H^(X, \mathbb{Q}))$ is a subgroup of finite index in $\text{Mon}(X)$.*

Following an idea of Beauville, Markman had proved in previous work that the class of the diagonal in $X \times X$ can be expressed in terms of the Chern classes of the universal sheaf \mathcal{E} . This implies

Theorem 4.2.3.3 (Markman, cf. [Mar1], Cor.2). *The Künneth factors of the Chern classes of the universal sheaf \mathcal{E} generate $H^*(X, \mathbb{Q})$.*

Summary of results used in the sequel. We have seen that there are homomorphisms $\varphi_i : \tilde{H}(S, \mathbb{Q}) \rightarrow H^{2i}(X, \mathbb{Q})$ and $\psi : H^2(X, \mathbb{Q}) \rightarrow v^\perp$ with the following properties:

- i) The φ_i and ψ are induced by algebraic cycles on $S \times X$ resp. on $X \times S$.
- ii) The homomorphism φ_2 induces an isomorphism

$$v^\perp \rightarrow H^2(X, \mathbb{Q})$$

whose inverse is ψ .

- iii) There is a subgroup of finite index $N \subset \text{Mon}(X)$ such that the compositions

$$\eta_{i,0} := \varphi_i \circ \psi : H^2(X, \mathbb{Q}) \rightarrow H^{2i}(X, \mathbb{Q})$$

are N -equivariant. This follows from Theorem 4.2.3.1 and 4.2.3.2.

- iv) For $i \geq 2$, the classes $\varphi_i(v) \in H^{2i}(X, \mathbb{Q})$ are N -invariant. Again, this is implied by Theorem 4.2.3.1.

- v) The sum of $H^0(X, \mathbb{Q})$, of the image of $\oplus_{i \geq 1} \eta_{i,0}$ and of the $\varphi_i(v)$ generate the cohomology ring $H^*(X, \mathbb{Q})$ as a \mathbb{Q} -algebra. This is a consequence of Theorem 4.2.3.3.

4.2.4 The motive of X

Let $X = M_H(v)$ be as in the previous subsection, let Y be an algebraic deformation of X . By this we mean that there exists a smooth, projective morphism of connected, smooth, complex algebraic varieties $\mathcal{X} \rightarrow B$ which admits X and Y as fibers.

Denote by $\langle \mathfrak{h}^2(Y) \rangle$ the smallest Tannakian subcategory (see Definition 4.2.1.3) of \mathcal{M} containing $\mathfrak{h}^2(Y)$. Note that $\langle \mathfrak{h}^2(Y) \rangle$ contains the Tate motive since a polarization of Y induces an inclusion of the Lefschetz motive in $\mathfrak{h}^2(Y)$ and since the Tate motive is the dual of the Lefschetz motive. Now use that by definition $\langle \mathfrak{h}^2(Y) \rangle$ contains all subobjects of $\mathfrak{h}^2(Y)$ in \mathcal{M} . Moreover, by definition the motive $\mathbb{1} = \mathfrak{h}(\text{Spec}(\mathbb{C}))$ is an object of $\langle \mathfrak{h}^2(Y) \rangle$ the latter being a Tannakian subcategory of \mathcal{M} .

Theorem 5. *The André motive $\mathfrak{h}(Y)$ is an object of $\langle \mathfrak{h}^2(Y) \rangle$.*

Proof. The idea is to identify $\tilde{H}(S, \mathbb{Q})$ with $G(X) = H^2(X, \mathbb{Q}) \oplus \mathbb{Q}(-1)$ and to use Markman's results to define a surjection of a sum of products of $G(X)$ to $H^*(X, \mathbb{Q})$ which is monodromy invariant. By André's deformation

principle, this will induce a surjection of a motive $\mathfrak{m}(Y)$ to $\mathfrak{h}(Y)$ where $\mathfrak{m}(Y)$ is an object of $\langle \mathfrak{h}^2(Y) \rangle$. Let's make this precise now.

For any fibre V of $\mathcal{X} \rightarrow B$, let $\mathfrak{g}_0(V) := \mathfrak{h}^0(V) = \mathbb{1}$ and for $i = 1, \dots, d = \dim(X)$ define

$$\mathfrak{g}_i(V) := (\mathfrak{h}^2(V) \oplus \mathbb{1}(-1))(-i + 1).$$

(For $V = X = M_H(v)$, the motive $\mathfrak{g}_i(X)$ plays the role of $\tilde{\mathfrak{h}}(S) = \mathfrak{h}^0(S)(-1) \oplus \mathfrak{h}^2(S) \oplus \mathfrak{h}^4(S)(1)(-i + 1)$.)

Next, we put

$$\mathfrak{m}(V) := \bigoplus_{(i_1, \dots, i_d) \in \{0, \dots, d\}^d} (\mathfrak{g}_{i_1}(V) \otimes \dots \otimes \mathfrak{g}_{i_d}(V)).$$

Note that $\mathfrak{m}(V)$ is an object of $\langle \mathfrak{h}^2(V) \rangle$ and that $\mathfrak{m}(V)$ can be seen as a submotive of the motive of a variety Z which is a disjoint union of products of V and \mathbb{P}^1 .

We fix an isomorphism $\eta_0 : \mathbb{1} \rightarrow \mathfrak{h}^0(X)$. For $i = 1, \dots, d$ we will define below morphisms of motives

$$\eta_i : \mathfrak{g}_i(X) \rightarrow \mathfrak{h}^{2i}(X)$$

with the following properties:

- a) there exists a subgroup N of finite index in $\text{Mon}(X)$ such that η_i is N -invariant.
- b) if we define the morphism

$$\eta : \mathfrak{m}(X) \rightarrow \mathfrak{h}(X)$$

as the composition of the morphism

$$\bigoplus (\eta_{i_1} \otimes \dots \otimes \eta_{i_d}) : \mathfrak{m}(V) \rightarrow \bigoplus_{(i_1, \dots, i_d) \in \{0, \dots, d\}^d} (\mathfrak{h}^{2i_1}(X) \otimes \dots \otimes \mathfrak{h}^{2i_d}(X))$$

with the cup-product morphism

$$\bigoplus (\mathfrak{h}^{2i_1}(X) \otimes \dots \otimes \mathfrak{h}^{2i_d}(X)) \rightarrow \mathfrak{h}(X),$$

then η is surjective.

Assume for one moment, that the η_i are defined. Then by André's Theorem 4.2.2.5 applied to the family $\mathcal{Z} \rightarrow B$ which is constructed by letting vary the Z and to η , we get a surjection $\mathfrak{m}(Y) \rightarrow \mathfrak{h}(Y)$. Since $\mathfrak{m}(Y)$ is an object of $\langle \mathfrak{h}^2(Y) \rangle$ and since this category is closed under quotients in \mathcal{M} , the proof is reduced to the construction of the η_i .

Let $\eta_{i,0} : \mathfrak{h}^2(X)(-i + 1) \rightarrow \mathfrak{h}^{2i}(X)$ be the morphism of motives corresponding to the algebraic homomorphism $\eta_{i,0}$ in item iii) in the summary

at the end of the last section. Next, we define $\eta_{i,1} : \mathbb{1}(-i) \rightarrow \mathfrak{h}^{2i}(X)$ by $\eta_{i,1} = \varphi_i(v) \in A_{\text{mot}}^{2i}(X)$. Finally we define

$$\eta_i := \eta_{i,0} \oplus \eta_{i,1} : \mathfrak{g}_i(X) \rightarrow \mathfrak{h}^{2i}(X).$$

Property a) has been checked in items iii) and iv) at the end of the preceding section.

Property b) is a direct consequence of item v). There we have seen that the Betti realization of η is surjective. It remains to apply Theorem 4.2.2.4 which says that the Betti realization is exact and conservative, and hence that η is surjective in \mathcal{M} . \square

Using the results of [An2], this has some interesting consequences.

Corollary 4.2.4.1. *i) The motive of Y is an object of $\mathcal{M}(\text{Ab})$ which is the full, Tannakian subcategory of \mathcal{M} generated by the motives of Abelian varieties.*

ii) All Hodge classes on Y are motivated, in particular they are all absolute in the sense of Deligne.

iii) If the standard conjecture $B(V)$ holds for all smooth, projective varieties V , then the Hodge conjecture holds for Y .

Proof. i) is a direct consequence of [An2], Theorem 1.5.1. This theorem says that the motive $\mathfrak{h}^2(Y)$ of an irreducible symplectic, projective variety Y is an object of $\mathcal{M}(\text{Ab})$. The proof of this theorem relies once more on the Kuga–Satake correspondence. André shows that the Kuga–Satake homomorphism $P^2(Y) \hookrightarrow H^2(A \times A, \mathbb{Q})$ is motivated where $P^2(Y)$ is the primitive part of $H^2(Y, \mathbb{Q})$ with respect to some polarization and A is a Kuga–Satake variety for $P^2(Y)$. Thus, $\mathfrak{p}^2(Y)$, the motive corresponding to $P^2(Y)$, and hence also $\mathfrak{h}^2(Y)$ are objects of $\mathcal{M}(\text{Ab})$.

ii) follows from i) and from [An1], Theorem 0.6.2, which says that all Hodge classes on Abelian varieties are motivated. The proof of this theorem uses the deformation principle to reduce first to Abelian varieties with CM, then to Weil classes and finally to products of elliptic curves.

iii) is a direct consequence of ii) and the definition of a motivated class. \square

Bibliography

- [Ab1] S. Abdulali, *Algebraic cycles in families of abelian varieties*, Canad. J. Math. **46**, (1994), 1121-1134.
- [Ab2] S. Abdulali, *Abelian varieties of type III and the Hodge conjecture*, Int. J. Math. **10**, no. 6, (1999), 667-675.
- [AK] A. Altman, S. Kleiman, *Foundations of the theory of Fano schemes*, Comp. Math. **34** (1977), 3-47.
- [An1] Y. André, *Pour une théorie inconditionnelle des motifs*, Publ. Math. I.H.E.S. **83** (1996), 5-49.
- [An2] Y. André, *On the Shafarevich and Tate conjectures for hyperkähler varieties*, Math. Ann. **305** (1996), 205-248.
- [An3] Y. André, *Une introduction aux motifs*, Panoramas et Synthèses 17, SMF (2004).
- [An4] Y. André, *Déformation et spécialisation de cycles motivés*, J. Inst. Math. Jussieu **5**, issue 4 (2006), 563-603.
- [Ar] D. Arapura, *Motivation for Hodge cycles*, Adv. Math. **207**, no. 2 (2006), 762-781.
- [Be] A. Beauville, *Variétés Kählériennes dont la première classe de Chern est nulle*, J. Diff. Geom. **18** (1983), 755-782.
- [BD] A. Beauville, R. Donagi, *La variété des droites d'une hypersurface cubique de dimension 4*, C.R. Acad. Sci. Paris **301** (1982), 703-706.
- [BL] C. Birkenhake, H. Lange, *Complex abelian varieties. Second edition*, Grundlehren, Band 302, Springer (2004).
- [Bo] C. Borcea, *K3 surfaces and complex multiplication*, Rev. Roumaine Math. Pures Appl. **31** (1986), 499-505.
- [BHN] R. Brauer, H. Hasse, E. Noether, *Beweis eines Hauptsatzes in der Theorie der Algebren*, J. reine angew. Math. **167** (1932), 399-404.

- [BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron models*, Erg. Math. 3. Folge, Band 21, Springer (1990).
- [CDK] E. Cattani, P. Deligne, A. Kaplan, *On the locus of Hodge classes*, J. Amer. Math. Soc. **8** (1995), 483-506.
- [CM] A. Conte, J. Murre, *The Hodge conjecture for fourfolds admitting a covering by rational curves*, Math. Ann. **238** (1978), 79-88.
- [Da] G. Danila, *Sur la cohomologie d'un fibré tautologique sur le schéma de Hilbert d'une surface*, J. Alg. Geom. **10** (2001), 247-280.
- [De] P. Deligne, *Hodge cycles on abelian varieties*, (notes by J.S. Milne), in: LNM 900, Springer (1982), 9-100.
- [Dr] P. Draxl, *Skew fields*, LMS Lect. Note Series 31, Cambridge University Press (1983).
- [Fa] G. Fano, *Sul sistema ∞^2 di rette contenute in una varietà cubica generale dello spazio a quattro dimensioni*, Atti R. Acc. Sci. Torino **38** (1904), 778-792.
- [Fu] W. Fulton, *Intersection theory*, Erg. Math., 3. Folge, Band 2, 2nd ed., Springer (1998).
- [FG] B. Fantecchi, L. Göttsche, *Local properties and Hilbert schemes of points*, in: Fundamental algebraic geometry, Math. Surveys Monogr. 123, AMS (2005), 139-178.
- [vG1] B. van Geemen, *Theta functions and cycles on some abelian fourfolds*, Math. Zeit. **221** (1996), 617-631.
- [vG2] B. van Geemen, *An introduction to the Hodge conjecture for Abelian varieties*, in: Algebraic Cycles and Hodge Theory (Torino, 1993), LNM 1594, Springer (1994), 233-252.
- [vG3] B. van Geemen, *Kuga-Satake varieties and the Hodge conjecture*, in: The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), Kluwer, Dordrecht (2000), 51-82.
- [vG4] B. van Geemen, *Real multiplication on K3 surfaces and Kuga-Satake varieties*, Mich. Math. J. **56** (2008), 375-399.
- [vGV] B. van Geemen, A. Verra, *Quaternionic Pryms and Hodge classes*, Topology **42** (2003), 35-53.
- [GS] P. Gille, T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge studies Adv. Math. 101, Cambridge University Press (2006).

- [Go] B. Gordon, *A Survey of the Hodge Conjecture for Abelian Varieties*, Appendix B in: J. Lewis, *A Survey of the Hodge Conjecture*, CRM Monograph Series 10, 2nd ed. (1999).
- [GH] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Wiley Classics Lib., Wiley and Sons, New York (1994).
- [GHJ] M. Gross, D. Huybrechts, D. Joyce, *Calabi–Yau manifolds and related geometries*, Springer Universitext (2002).
- [Gr] A. Grothendieck, *Sur quelques points d’algèbre homologique*, Tôhoku Math. J. **9** (1957), 119-221.
- [Har] R. Hartshorne, *Algebraic geometry*, GTM 52, Springer, New York (1977).
- [Has] B. Hassett, *Special cubic fourfolds*, Comp. Math. **120** (2000), 1-23.
- [Ho] W. Hodge, *The theory and applications of harmonic integrals*, Cambridge University Press (1941).
- [Hu] D. Huybrechts, *Moduli spaces of hyperkähler manifolds and mirror symmetry*, in: ICTP Lecture Notes XIX, Trieste (2004), 185-247.
- [K3] *Géométrie des surfaces K3: modules et périodes*, editors: A. Beauville, J.P. Bourguignon, M. Demazure, Astérisque **126** (1985).
- [Kl] S. Kleiman, *Algebraic cycles and the Weil conjectures*, in: Dix exposés sur la cohomologie des schémas, North-Holland, Amsterdam (1968), 359-386.
- [KS] M. Kuga, I. Satake, *Abelian varieties attached to polarized K3 surfaces*, Math. Ann. **169** (1967), 239-242.
- [L] S. Lang, *Algebra*, 3rd ed., Addison-Wesley (1993).
- [La] R. Laterveer, *Algebraic varieties with small Chow groups*, J. Math. Kyoto Univ. **38** (1998), 673-694.
- [Lo] G. Lombardo, *Abelian varieties of Weil type and Kuga–Satake varieties*, Tôhoku Math. J. **53** (2001), 453-466.
- [Mar1] E. Markman, *Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces*, J. reine angew. Math. **544** (2002), 61-82.
- [Mar2] E. Markman, *On the monodromy of moduli spaces of sheaves on K3 surfaces*, J. Alg. Geom. **17** (2008), 29-99.

- [Mat] A. Mattuck, *Cycles on abelian varieties*, Proc. Amer. Math. Soc. **9** (1958), 88-98.
- [Mi] E. Mistretta, *Some constructions around stability of vector bundles on projective varieties*, PhD Thesis, Univ. Paris VII (2006).
- [Mo] D. Morrison, *Algebraic cycles on products of surfaces*, Proc. Algebraic Geometry Symposium, Tôhoku University (1984), 194-210.
- [MSY] K. Matsumoto, T. Sasaki, M. Yoshida, *The monodromy of the period map of a 4-parameter family of K3 surfaces and the hypergeometric function of type (3,6)*, Int. J. Math. **3** (1992), 1-164.
- [Mu1] S. Mukai, *On the moduli space of bundles on K3 surfaces, I*, in: Vector Bundles on Algebraic Varieties, Bombay (1984), 341-413.
- [Mu2] S. Mukai, *Vector bundles on a K3 surface*, Proc. of the ICM, Vol. II, (Beijing, 2002), 495-502.
- [MZ] B. Moonen, Y. Zarhin, *Hodge classes on abelian varieties of low dimension*, Math. Ann. **315** (1999), 711-733.
- [Mum] D. Mumford, *Prym varieties I*, in: Contributions to analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York (1974), 325-350.
- [N] V. Nikulin, *On correspondences between surfaces of K3 type*, (in Russian), translation in: Math. USSR-Izv. **30** (1988), 375-383.
- [NW] M. Nieper-Wisskirchen, *Is c_2 a multiple of the Beauville–Bogomolov form?*, available at <http://www.math.uni-augsburg.de/alg/mitarbeiter/mnieper/arbeiten>
- [O'G] K. O'Grady, *The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface*, J. Alg. Geom. **6** (1997), 599-644.
- [O] K. Oguiso, *Local families of K3 surfaces and applications*, J. Alg. Geom. **12** (2003), 405-432.
- [P] K. Paranjape, *Abelian varieties associated to certain K3 surfaces*, Comp. Math. **68** (1988), 11-22.
- [PSS] I. Piatetski-Shapiro, I. Shafarevich, *The arithmetic of K3 surfaces*, Proc. Steklov Inst. Math. **132** (1973), 45-57.
- [RM] J. Ramón-Marí, *On the Hodge conjecture for products of certain surfaces*, eprint [math.AG/055357](https://arxiv.org/abs/math/055357).
- [Ri] J. Rizov, *Complex multiplication for K3 surfaces*, eprint [math.AG/0508018](https://arxiv.org/abs/math/0508018).

- [Sch] C. Schoen, *Hodge classes on self-products of a variety with an automorphism*, Comp. Math. **65** (1988), 3-32.
- [Se] J.-P. Serre, *A course in arithmetic*, GTM 7, Springer, New-York (1973).
- [Sha] I. Shafarevich, *Le théorème de Torelli pour les surfaces algébriques de type K3*, in: Actes Congrès intern. math. (1970), 413-417.
- [Shi] T. Shioda, *The Hodge conjecture for Fermat varieties*, Math. Ann. **245** (1979), 175-184.
- [SI] T. Shioda, H. Inose, *On singular K3 surfaces*, in: Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo (1977), 119-136.
- [Ta] J. Tate, *Algebraic cycles and poles of zeta functions*, in: Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ. 1963), Harper & Row, New York (1965), 93-110.
- [Ti] J.-P. Tignol, *On the Corestriction of Central Simple Algebras*, Math. Zeit. **194** (1987), 267-274.
- [Tk] S. Tankeev, *Cycles on simple abelian varieties of prime dimension*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 155-170.
- [Ve1] M. Verbitsky, *Tri-analytic subvarieties of hyper-Kähler manifolds*, Geom. Funct. Anal. **5** (1995), 92-104.
- [Ve2] M. Verbitsky, *Hyperholomorphic bundles over a hyper-Kähler manifold*, J. Alg. Geom. **5** (1996), 633-669.
- [Vo1] C. Voisin, *Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes*, in: Complex projective geometry (Trieste 1989), LMS Lect. Note Series 179, 294-303.
- [Vo2] C. Voisin, *Théorie de Hodge et géométrie algébrique complexe*, Cours Spécialisés 10, SMF, Paris (2002).
- [Vo3] C. Voisin, *Some aspects of the Hodge conjecture*, Japan. J. Math. **2** (2007), 261-296.
- [Vo4] C. Voisin, *On the Chow ring of certain algebraic hyper-Kähler manifolds*, Pure Appl. Math. Quart. **4** (2008), 613-649.
- [Z] Y. Zarhin, *Hodge groups of K3 surfaces*, J. reine angew. Math. **341** (1983), 193-220.