Chapter 2
Continuous Optimization in Computer Vision

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Overview

1. Introduction
2. Convex envelopes
3. Convex optimization
4. A class of convex problems
5. Algorithms
6. Implementation and applications
Energy minimization methods

• Typical energies consist of a regularization term and a data term

\[ \min_u \{ E(u) = \mathcal{R}(u) + \mathcal{D}(u, f) \}, \]

where \( f \) is the input data and \( u \) is the unknown solution

• Energy functional is designed such that low-energy states reflect the physical properties of the problem

• Minimizer provides the best (in the sense of the model) solution to the problem
Energy minimization methods

- **Discrete MRF setting:** Images are represented as graphs $G(\mathcal{V}, \mathcal{E})$, consisting of a node set $\mathcal{V}$, and an edge set $\mathcal{E}$. Each node $v \in \mathcal{V}$ can take a label from a discrete label set $\mathcal{U} \subset \mathbb{Z}$, i.e. $u(v) \in \mathcal{U}$.

- **Continuous Variational setting:** Images are considered as continuous functions $u : \Omega \rightarrow \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is the image domain.

- **Link to statistics:** In a Bayesian setting, the energy relates to the posterior probability via a Gibbs distribution

\[
p(u|f) = \frac{1}{Z} \exp(-E(u))
\]

- Computing the minimizer of $E(u)$ is equivalent to MAP estimation on $p(u|f)$.
The posterior distribution

- $p(u|f)$ provides a complete distribution.
- Which sample should we selected?
- Expectation: Compute sample, that minimizes the squared distance to the distribution

$$\bar{u} = \frac{1}{\mathcal{Z}} \int u p(u|f) \, du$$

- Needs subtle algorithms to approximate the integral (MCMC)
- MAP: Computing that sample that maximizes the probability

$$u^* = \arg \max_u p(u|f) = \arg \min_u E(u)$$

- Leads to well-defined optimization algorithms
Example: Total variation based image restoration

Image model: \( f = k \ast u + n \), blur kernel \( k = \)

Variational model: [Rudin, Osher, Fatemi ’92]

\[
\min_u \int_{\Omega} |Du| + \frac{\lambda}{2} \|k \ast u - f\|^2_2
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(a) Degraded image \( f \)
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(a) Degraded image \( f \)  
(b) Reconstructed image \( u \)

We need to solve large scale optimization problems
Consider the following general mathematical optimization problem:

$$\min f_0(x)$$

$$\text{s.t. } f_i(x) \leq 0, \quad i = 1 \ldots m$$

$$x \in S,$$

where $f_0(x) \ldots f_m(x)$ are real-valued functions, $x = (x_1, \ldots x_n)^T \in \mathbb{R}^n$ is a $n$-dimensional real-valued vector, and $S$ is a subset of $\mathbb{R}^n$.

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How to solve this problem?

- Naive: “Download a commercial package ...”
Optimization problems are unsolvable

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How to solve this problem?

- Naive: “Download a commercial package ...”
- Reality: “Finding a solution is far from being trivial!”
- “Optimization problems are unsolvable”

[Nesterov ’04]
Consider the problem class $C_0$ [Nesterov '04]

$$\min_{x \in B_n} f(x),$$

where $B_n$ is the $n$-dimensional unit box defined by

$$B_n = \{x \in \mathbb{R}^n : 0 \leq x_i \leq 1, \; i = 1...n\}$$
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What is the lower complexity bound to find an $\varepsilon$-approximate solution?
A remark on Lipschitz continuity: A function $f(x)$ is called Lipschitz continuous on $\mathcal{B}_n$ if

$$|f(x) - f(y)| \leq L \|x - y\|_\infty, \quad \forall x, y \in \mathcal{B}_n,$$

where the constant $L$ is an upper bound to the maximum steepness of $f(x)$.
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- **Theorem 1.1.2 [Nesterov ’04]**
  For $\frac{1}{2} L > \varepsilon > 0$, the complexity of a zero-order method to find an $\varepsilon$ - approximate solution of the problem class $C_0$ is at least

$$\left(\left\lfloor \frac{L}{2\varepsilon} \right\rfloor \right)^n$$
• Consider a general optimization problem with $n = 10$ unknowns and a moderate Lipschitz constant of $L = 2$
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Example

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• Complexity bounds for higher order methods (gradient descend, Newton, \ldots) are not much better
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Comparison to NP-hard problems: Hard combinatorial problems need \( 2^n \) arithmetic operations (only)!
Convex versus non-convex

- Non-convex problems
  - Often give more accurate models
  - In general no chance to find the global minimizer
  - Result strongly depends on the initialization
  - Dilemma: Wrong model or wrong algorithm?
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  - Note: Convex does not mean easy!
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- Current research: Bridging the gap between convex and non-convex optimization
  - Convex approximations of non-convex models
  - New models
  - Algorithms
  - Bounds
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The convex conjugate

- The convex conjugate $f^*(y)$ of a function $f(x)$ is defined through the Legendre-Fenchel transform

$$f^*(y) = \sup_{x \in \text{dom } f} \langle x, y \rangle - f(x)$$

- $f^*(y)$ is a convex function (pointwise supremum over linear functions)
Examples

- $f(x) = |x|$: 

\[ f^*(y) = \sup_x \langle x, y \rangle - |x| = \begin{cases} 0 & \text{if } |y| \leq 1 \\ \infty & \text{else} \end{cases} \]

- $f(x) = \frac{1}{2} x^T Q x$, $Q$, positive definite

\[ f^*(y) = \sup_x \langle x, y \rangle - \frac{1}{2} x^T Q x = \frac{1}{2} y^T Q^{-1} y \]
The convex envelope

- The biconjugate function is defined by twice application of the Legendere-Fenchel transform

$$f^{**}(x) = \sup_{y \in \text{dom } f^*} \langle x, y \rangle - f^*(y)$$
The convex envelope

- The biconjugate function is defined by twice application of the Legendere-Fenchel transform
  \[ f^{**}(x) = \sup_{y \in \text{dom } f^*} \langle x, y \rangle - f^*(y) \]

- The biconjugate function \( f^{**}(x) \) is the largest convex l.s.c. function below \( f(x) \)
- In case of uniqueness, \( f(x) \) and \( f^{**}(x) \) have the same minimum
- If \( f(x) \) is a convex, l.s.c. function, \( f^{**}(x) = f(x) \)

- Computing the biconjugate function allows to “convexify” a non-convex problem
- Unfortunately, computing \( f^{**}(x) \) is not tractable for most problems
Approximate envelopes

• It is not possible to compute $f^{**}$ in general (there are exceptions, though)
• What about approximate envelopes?
• Many problems can be decomposed as

$$f(x) = \sum_{i=1}^{n} f_i(x)$$

where $f_i^{**}(x)$ is often much easier to compute than $f^{**}(x)$.

• It can be shown that

$$f(x) > f^{**}(x) = \left( \sum_i f_i \right)^{**}(x) > \left( \sum_i f_i^{**} \right)(x)$$

• This is exactly, what current research is about
• State-of-the-art techniques (LP, QBPO, graph cuts, convex relaxations ...) exactly try to approximate the convex envelope in this way
Consider the problem

$$\min_x f_1(x) - f_2(x),$$

where $f_1, f_2$ are real-valued functions.

Classical form of duality developed by W. Fenchel in the 40s.
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• Consider the equivalent constrained problem

$$\min_{x_1, x_2} f_1(x_1) - f_2(x_2), \quad \text{s.t. } x_1 = x_2$$
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$$\min_{x_1, x_2} f_1(x_1) - f_2(x_2), \quad \text{s.t. } x_1 = x_2$$

• Introducing Lagrange multipliers $y$ for the constraint $x_1 = x_2$ yields

$$\min_{x_1, x_2} \max_y f_1(x_1) - f_2(x_2) + \langle x_1 - x_2, y \rangle$$
Fenchel-Duality

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  \]
- We can recognize the definition of the convex conjugate functions, leads to the dual problem
  \[
  \max_y f_2^*(y) - f_1^*(y)
  \]
Fenchel’s duality theorem

If $f_1$ convex and $f_2$ concave, then

$$
\min_x f_1(x) - f_2(x) = \max_y f_2^*(y) - f_1^*(y)
$$

Interpretation: Minimizing the vertical distance between two points on the curves is equivalent to maximizing the distance between two parallel tangents on the curves.
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A general convex optimization problem is defined as

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\min & \quad f_0(x) \\
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& \quad Ax = b
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where \( f_0(x) \ldots f_p(x) \) are real-valued convex functions, 
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Convex optimization problems are considered as solvable!

“The great watershed in optimization isn’t between linearity and non-linearity, but convexity and non-convexity” [Rockafellar ’93]
Black box convex optimization

Generic iterative algorithm for convex optimization:

1. Pick any initial vector $\mathbf{x}^0 \in \mathbb{R}^n$, set $k = 0$
2. Compute search direction $\mathbf{d}^k \in \mathbb{R}^n$
3. Choose step size $\tau^k$ such that $f(\mathbf{x}^k + \tau^k \mathbf{d}^k) < f(\mathbf{x}^k)$
4. Set $\mathbf{x}^{k+1} = \mathbf{x}^k + \tau^k \mathbf{d}^k$, $k = k + 1$
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Different methods to determine the search direction $d^k$

- steepest descend
- conjugate gradients
- Newton, quasi-Newton
- Working-horse for the lazy: Limited memory BFGS quasi-Newton method [Nocedal '80]
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Different methods to determine the search direction $d^k$

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Black box methods do not exploit the structure of the problem and hence are often less effective
Structured convex optimization

There are a number of efficient solvers available for important types of structured convex optimization problems [Boyd, Vandenberghe ’04]:
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- Least squares problems:

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\min_x ||Ax - b||_2^2
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There are a number of efficient solvers available for important types of structured convex optimization problems [Boyd, Vandenberghe ’04]:

- Least squares problems:
  \[ \min_x \|Ax - b\|_2^2 \]

- Linear program (LP)
  \[ \min c^T x, \quad \text{s.t. } Ax = b, \quad x \geq 0 \]
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- **Quadratic program (QP)**
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  \min \frac{1}{2} x^T Q x + c^T x, \ Q \succeq 0 \\
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• ...

Many problems can be formulated in these frameworks

• Fast methods available (simplex, interior point, ...)

• Sometimes ineffective for large-scale problems
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Let us consider the following class of structured convex optimization problems

$$\min_{x \in X} F(Kx) + G(x),$$

- $K : X \rightarrow Y$ is a linear and continuous operator from a Hilbert space $X$ to a Hilbert space $Y$.
- $F : Y \rightarrow \mathbb{R} \cup \{\infty\}, G : X \rightarrow \mathbb{R} \cup \{\infty\}$ are “simple” convex, proper, l.s.c. functions, and hence have an easy to compute prox operator:

$$\text{prox}_G(z) = (I + \partial G)^{-1}(z) = \arg \min_x \frac{\|x - z\|^2}{2} + G(x)$$
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- It turns out that many standard problems can be cast in this framework.
- There exists a vast literature of numerical algorithms to solve this class of problems.
Examples

- **Image restoration: The ROF model**

  \[
  \min_u \| \nabla u \|_1 + \frac{\lambda}{2} \| k * u - f \|_2^2 ,
  \]

- **Compressed sensing: Basis pursuit problem (LASSO)**

  \[
  \min_x \| x \|_1 + \frac{\lambda}{2} \| Ax - b \|_2^2
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- Machine learning: Linear support vector machine
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  \min_{w,b} \frac{\lambda}{2} \|w\|_2^2 + \sum_{i=1}^{n} \max(0, 1 - y_i (\langle w, x_i \rangle + b))
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Examples

- Image restoration: The ROF model
  \[ \min_u \| \nabla u \|_1 + \frac{\lambda}{2} \| k * u - f \|_2^2, \]

- Compressed sensing: Basis pursuit problem (LASSO)
  \[ \min_x \| x \|_1 + \frac{\lambda}{2} \| A x - b \|_2^2 \]

- Machine learning: Linear support vector machine
  \[ \min_{w, b} \frac{\lambda}{2} \| w \|_2^2 + \sum_{i=1}^{n} \max (0, 1 - y_i (\langle w, x_i \rangle + b)) \]

- General linear programming problems
  \[ \min_x \langle c, x \rangle, \quad \text{s.t.} \begin{cases} Ax = b \\ x \geq 0 \end{cases} \]
The real power of convex optimization comes through duality.

Recall the convex conjugate:

\[ F^*(y) = \sup_{x \in X} \langle x, y \rangle - F(x), \quad F^{**}(x) = \sup_{y \in Y} \langle x, y \rangle - F^*(y) \]

If \( F(x) \) convex, l.s.c. than \( F^{**}(x) = F(x) \)
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\min_{x \in X} \max_{y \in Y} \langle Kx, y \rangle + G(x) - F^*(y) \quad \text{(Primal-Dual)}
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A class of convex problems

Primal, dual, primal-dual

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\]

\[
\max_{y \in Y} - (F^*(y) + G^*(-K^*y)) \quad \text{(Dual)}
\]

Allows to compute the duality gap, which is a measure of optimality
Optimality conditions

We focus on the primal-dual formulation:

$$\min_{x \in X} \max_{y \in Y} \langle Kx, y \rangle + G(x) - F^*(y)$$

We assume, there exists a saddle-point \((\hat{x}, \hat{y}) \in X \times Y\) which satisfies the Euler-Lagrange equations

$$\begin{cases} K\hat{x} - \partial F^*(\hat{y}) & \geq 0 \\ K^*\hat{y} + \partial G(\hat{x}) & \geq 0 \end{cases}$$
A note on Forward-backward splitting

- Consider the problem $\min_x f_1(x) + f_2(x)$, where $f_1(x)$ is a convex function and $f_2(x)$ is a convex function with $L$-Lipschitz continuous gradient $\nabla f_2$, i.e.
  \[
  \|\nabla f_2(x) - \nabla f_2(y)\| \leq L\|x - y\|, \forall x, y \in \text{dom}(f_2)
  \]

- It can be shown that a minimizer can be characterized by the fixed point equation [Combettes, Pesquet '05]
  \[
  x = (I + \tau \partial f_1)^{-1}(x - \tau \nabla f_2(x))
  \]

- This suggests the forward-backward iterations
  \[
  x^{n+1} = (I + \tau \partial f_1)^{-1}(x^n - \tau \nabla f_2(x^n))
  \]

- **Note:** This is exactly the case for our saddle-point formulation
  \[
  \min_x \max_y \langle Kx, y \rangle + G(x) - F^*(y)
  \]
Arrow-Hurwicz algorithm

- Classical method for finding saddle-points [Arrow, Hurwicz, Uzawa ’58]

\[
\begin{align*}
    x^{n+1} &= (I + \tau \partial G)^{-1}(x^n - \tau K^* y^n) \\
    y^{n+1} &= (I + \tau \partial F^*)^{-1}(y^n + \tau Kx^{n+1})
\end{align*}
\]

- Alternating forward-backward step in the primal variable and the dual variable
- Convergence under quite restrictive assumptions on $\tau$
- For some problems very fast, e.g. ROF problem using adaptive time steps [Zhu, Chan, ’08]
Proximal point algorithm

- Very general algorithm with well-studied properties [Martinet ’70], [Rockafellar ’76]
- Searches the zero of an operator, i.e. $0 \in T$
- General algorithm

$$p^{n+1} = (I + \tau^n T)^{-1} p^n$$

- In our setting:

$$p = \begin{pmatrix} x \\ y \end{pmatrix}, \quad T = \begin{pmatrix} \partial G & K^* \\ -K & \partial F^* \end{pmatrix}$$

- “Not implementable” since in most interesting cases, $T$ is very hard to invert
- Many “implementable” algorithms can be shown to be a special case of the proximal point algorithm
Other interesting algorithms

- Extragradient methods [Korpelevich ’76], [Popov ’80]
  - Similar to Arrow-Hurwicz, but twice evaluation of the proximal operators
- Algorithms proposed by Yurii Nesterov
  - Many different algorithms, with guaranteed convergence rates, e.g. [Nesterov ’03]
  - Recently, an increased interest in signal processing, computer vision and machine learning
- Douglas-Rachford splitting (DRS): [Mercier, Lions ’79]
  - Can be applied if the problem can be decomposed into simpler problems
  - Special case of the proximal-point algorithm [Eckstein, Bertsekas ’89]
  - Equivalent to an alternating minimization (ADMM) on the augmented Lagrangian formulation [Eckstein, Bertsekas ’89]
  - Equivalent to the split-Bregman algorithm [Goldstein, Osher ’09]
Other interesting algorithms

- Iterative shrinkage thresholding algorithm (ISTA) [Daubechies, Defrise, De Mol ’04]
  - Developed for sparse recovery
  - Fast iterative shrinkage thresholding algorithm (FISTA) [Beck, Teboulle ’09]
- Proximal splitting algorithms, e.g. [Combettes, Pesquet, ’08]
  - Many different algorithms specialized to interesting problems in image processing and sparse recovery
A simple primal-dual algorithm

Proposed in a series of papers: [Pock, Cremers, Bischof, Chambolle, ’09], [Chambolle, Pock, ’10], [Pock, Chambolle, ’11]

- Initialization: Choose s.p.d. $T$, $\Sigma$, $\theta \in [0, 1]$, $(x^0, y^0) \in X \times Y$.
- Iterations ($n \geq 0$): Update $x^n, y^n$ as follows:

$$
\begin{align*}
  x^{n+1} &= (I + T\partial G)^{-1}(x^n - TK^*y^n) \\
  y^{n+1} &= (I + \Sigma\partial F^*)^{-1}(y^n + \Sigma K(x^{n+1} + \theta(x^{n+1} - x^n)))
\end{align*}
$$

- Alternates gradient descend in $x$ and gradient ascend in $y$
- Linear extrapolation of iterates of $x$ in the $y$ step
- $T$, $\Sigma$ can be seen as preconditioning matrices
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$$

• Alternates gradient descend in $x$ and gradient ascend in $y$
• Linear extrapolation of iterates of $x$ in the $y$ step
• $T$, $\Sigma$ can be seen as preconditioning matrices

• Can be derived from a pre-conditioned ADMM algorithm
• Can be seen as a relaxed Arrow-Hurwicz scheme
• Can be seen as an approximate extragradient scheme
The iterations of PD can be written as the variational inequality [He, Yuan ’10]

\[
\left\langle \begin{pmatrix} x - x^{n+1} \\ y - y^{n+1} \end{pmatrix}, \begin{pmatrix} \partial G(x^{n+1}) + K^* y^{n+1} \\ \partial F^*(y^{n+1}) - K x^{n+1} \end{pmatrix} \rightangle + M \begin{pmatrix} x^{n+1} - x^n \\ y^{n+1} - y^n \end{pmatrix} \geq 0,
\]

\[
M = \begin{bmatrix} T^{-1} & -K^* \\ -\theta K & \Sigma^{-1} \end{bmatrix}
\]
Relations to the proximal-point algorithm

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where

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This is exactly the proximal-point algorithm, but with a norm in $M$.

Convergence is ensured if $M$ is symmetric and positive definite.
Relations to the proximal-point algorithm

• The iterations of PD can be written as the variational inequality [He,Yuan ’10]

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\left\langle \left( \begin{array}{c} x - x^{n+1} \\ y - y^{n+1} \end{array} \right), \left( \begin{array}{c} \partial G(x^{n+1}) + K^* y^{n+1} \\ \partial F^*(y^{n+1}) - K x^{n+1} \end{array} \right) \right\rangle \geq 0,
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\]

• This is exactly the proximal-point algorithm, but with a norm in \(M\).

• Convergence is ensured if \(M\) is symmetric and positive definite.

• This is the case if \(\theta = 1\) and the assertion \(\|\Sigma^{1/2} K T^{1/2}\|^2 < 1\) is fulfilled.
A family of diagonal preconditioners

- It is important to choose $T, \Sigma$ such that the prox-operators are still easy to compute
- Restrict the preconditioning matrices to diagonal matrices
- It turns out that: Let $K \in \mathbb{R}^{m \times n}$, $T = \text{diag}(\tau)$ and $\Sigma = \text{diag}(\sigma)$ such that
  \[
  \tau_j = \frac{1}{\sum_{i=1}^{m} |K_{i,j}|^{2-\alpha}}, \quad \sigma_j = \frac{1}{\sum_{j=1}^{n} |K_{i,j}|^{\alpha}}
  \]
  then for any $\alpha \in [0, 2]$
  \[
  \| \Sigma^{\frac{1}{2}} K T^{\frac{1}{2}} \|^2 \leq 1.
  \]

- Gives an automatic problem-dependent choice of the primal and dual steps
- Allows to apply the algorithm in a plug-and-play fashion
- For $\alpha = 0$, equivalent to the alternating step method [Eckstein, Bertsekas, '90]
The algorithm gives different convergence rates on different problem classes [Chambolle, Pock, ’10]

- $F^*$ and $G$ nonsmooth: $O\left(1/N\right)$
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- $F^*$ or $G$ uniformly convex: $O(1/N^2)$
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- $F^*$ and $G$ uniformly convex: $O(\omega^N)$, $\omega < 1$
- Coincides with so far best known rates of first-order methods
Parallel computing?

The algorithm basically computes matrix-vector products
The matrices are usually very sparse
Parallel computing?

The algorithm basically computes matrix-vector products
The matrices are usually very sparse

- One simple processor for each unknown
- Each processor has a small amount of local memory
- According to the structure of $K$, each processor can inter-change data with its neighboring processors
Parallel computing?

The algorithm basically computes matrix-vector products. The matrices are usually very sparse:

- One simple processor for each unknown
- Each processor has a small amount of local memory
- According to the structure of $K$, each processor can inter-change data with its neighboring processors

Recent GPUs already go into this direction.
## Overview

1. Introduction
2. Convex envelopes
3. Convex optimization
4. A class of convex problems
5. Algorithms
6. Implementation and applications
We consider a regular Cartesian grid of size $M \times N$:

$$
\Omega_h = \{(ih, jh) : 1 \leq i \leq M, 1 \leq j \leq N\},
$$

where $h$ denotes the size of the spacing and $(i, j)$ denote the indices of the discrete locations $(ih, jh)$ in the image domain.
The gradient $\nabla u$ is a vector in the vector space $Y = X \times X$. For discretization of $\nabla : X \to Y$, we use standard finite differences with Neumann boundary conditions

$$(\nabla u)_{i,j} = \left( \begin{array}{c} (\nabla u)^1_{i,j} \\ (\nabla u)^2_{i,j} \end{array} \right),$$

where

$$(\nabla u)^1_{i,j} = \begin{cases} \frac{u_{i+1,j} - u_{i,j}}{h} & \text{if } i < M \\ 0 & \text{if } i = M \end{cases}, \quad (\nabla u)^2_{i,j} = \begin{cases} \frac{u_{i,j+1} - u_{i,j}}{h} & \text{if } j < N \\ 0 & \text{if } j = N \end{cases}.$$

Furthermore we will also need the discrete divergence operator $\text{div} \, p : Y \to X$, which is chosen to be adjoint to the discrete gradient operator. In particular, one has $-\text{div} = \nabla^*$ which is defined through the identity

$$\langle \nabla u, p \rangle_Y = -\langle u, \text{div} \, p \rangle_X.$$

Preconditioner: We easily find that $\Theta = \text{diag}(\tau)$, where $\tau = h/4$ and $\Sigma = \text{diag}(\sigma)$, where $\sigma = h/2$. 
The discrete ROF model

In the continuous setting, the ROF model is defined as

$$\min_{u} \int_{\Omega} |Du| + \frac{\lambda}{2} \|u - g\|_2^2$$

Using the discrete setting, we obtain

$$h^2 \min_{u \in X} \|\nabla u\|_1 + \frac{\lambda}{2} \|u - g\|_2^2,$$

where the discrete total variation is given by

$$\|\nabla u\|_1 = \sum_{i,j} |(\nabla u)_{i,j}|,$$
$$|(\nabla u)_{i,j}| = \sqrt{((\nabla u)_{i,j}^1)^2 + ((\nabla u)_{i,j}^2)^2},$$

and

$$\|u - g\|_2^2 = \sum_{i,j} (u_{i,j} - g_{i,j})^2$$

The parameter $\lambda$ is used to control the influence of regularization.
Casting the discrete ROF model in our framework, we obtain the following primal-dual ROF model

\[
\min_{u \in X} \max_{p \in Y} - \langle u, \text{div} p \rangle_X + \frac{\lambda}{2} \|u - g\|_2^2 - \delta_P(p),
\]

where the convex set \( P \) is given by

\[
P = \{ p \in Y : \|p\|_\infty \leq 1 \},
\]

and \( \|p\|_\infty \) denotes the discrete maximum norm defined as

\[
\|p\|_\infty = \max_{i,j} |p_{i,j}|, \quad |p_{i,j}| = \sqrt{(p_{i,j}^1)^2 + (p_{i,j}^2)^2}.
\]
Primal-dual algorithm to solve the ROF model

**ROF-Primal-Dual**

- **Initialization:** Choose $\tau = h/4$, $\sigma = h/2$, set $u^0 = \bar{u}^0 = f$, $p^0 = 0$, $\theta = 1$
- **Iterations ($n \geq 0$)**

\[
\begin{align*}
    u^{n+1} &= (I + \tau \partial G)^{-1}(u^n - \tau (-\text{div} p^n)) \\
    p^{n+1} &= (I + \sigma \partial F^*)^{-1}(p^n + \sigma \nabla (2u^{n+1} - u^n))
\end{align*}
\]
Finally, we need to find explicit expressions for \((I + \sigma \partial F^*)^{-1}\) and \((I + \tau \partial G)^{-1}\). We have that \(F^*(p) = \delta_P(p)\) and \(G(u) = \frac{\lambda}{2} \|u - g\|^2\). Since \(F^*\) is the indicator function of a convex set, the resolvent operator reduces to pointwise Euclidean projectors onto \(L^2\) balls.

\[
p = (I + \sigma \partial F^*)^{-1}(\tilde{p}) \iff p_{i,j} = \frac{\tilde{p}_{i,j}}{\max(1, |\tilde{p}_{i,j}|)}.
\]

The resolvent operator with respect to \(G\) poses simple pointwise quadratic problems. The solution is trivially given by

\[
u = (I + \tau \partial G)^{-1}(\tilde{u}) \iff u_{i,j} = \frac{\tilde{u}_{i,j} + \tau \lambda g_{i,j}}{1 + \tau \lambda}.
\]
Practical implementation

- Matlab implementation $\rightarrow$ Online-Demo
  - Uses Matlab vector operations
  - Takes a few seconds for typical images
  - About 100 lines of code
  - Files can be downloaded from the Tutorial homepage

- Parallel implementation $\rightarrow$ Online-Demo
  - GPU implementation using Nvidia CUDA
  - Matlab fronted
  - About 1000 lines of code
  - Depending on the GPU 100-1000 times faster

(a) Clean
(b) Noisy
(c) Denoised

ROF-denoising on the GPU
Linear programming

- Many applications of LP relaxations in computer vision and machine learning
- LP program in equality form

\[
\min c^T x \quad \text{s.t.} \quad Ax = b, \ x \geq 0, \]

(1)

- Preconditioned primal-dual algorithm

\[
\begin{align*}
x^{k+1} &= \text{proj}_{[0, \infty)} (x^k - T(A^T y^k + c)) \\
y^{k+1} &= y^k + \sum (A(2x^{k+1} - x^k) - b),
\end{align*}
\]

(2)

- Comparison
Graph cuts

- Graph cuts are widely used in computer vision [3]
- Can be written as a weighted total variation energy

\[
\min_u \|D_w u\|_{\ell_1} + \langle u, w^u \rangle, \text{ s.t. } 0 \leq u \leq 1, \quad (3)
\]

- Preconditioned primal-dual algorithm

\[
\begin{align*}
  u^{k+1} &= \text{proj}_{[0,1]} \left( u^k + T(D_w^T y^k - w^u) \right) \\
  y^{k+1} &= \text{proj}_{[-1,1]} \left( y^k + \Sigma(D_w(2u^{k+1} - u^k)) \right)
\end{align*}
\]

- Comparison

<table>
<thead>
<tr>
<th></th>
<th>MAXFLOW</th>
<th>PD</th>
<th>P-PD</th>
<th>P-PD-GPU</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.160s</td>
<td>15.75s</td>
<td>8.56s</td>
<td>0.045s</td>
</tr>
</tbody>
</table>
Continuous Potts model

- The Continuous Potts model [4] with $k$ phases can be written as a convex problem

$$\min_{u \in S} \max_{q \in B} \sum_{l=1}^{k} \langle Du_l, q_l \rangle + \langle u_l, f_l \rangle,$$

where $S$ is the simplex constraint and $B$ is an intersection of non-local $\ell_2$ balls.

- Comparison

<table>
<thead>
<tr>
<th></th>
<th>PD</th>
<th>P-PD</th>
<th>Speedup</th>
</tr>
</thead>
<tbody>
<tr>
<td>Synthetic (3 phases)</td>
<td>221.71s</td>
<td>75.65s</td>
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<tr>
<td>Synthetic (4 phases)</td>
<td>1392.02s</td>
<td>538.83s</td>
<td>2.6</td>
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<td>Natural (8 phases)</td>
<td>592.85s</td>
<td>113.76s</td>
<td>5.2</td>
</tr>
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</table>
Summary and open questions

- Energy minimization methods for computer vision
- Leads to large scale optimization problems
- Introduction into convex analysis and convex optimization
- Flexible primal-dual algorithm for a class of structured convex optimization problems
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- Energy minimization methods for computer vision
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- Flexible primal-dual algorithm for a class of structured convex optimization problems

- Can first-order methods go beyond the shown convergence rates?
- Combining first- and second-order methods?
- Discrete versus continuous optimization methods?