

Chapter 3

Variational Methods and Geometric Reconstruction



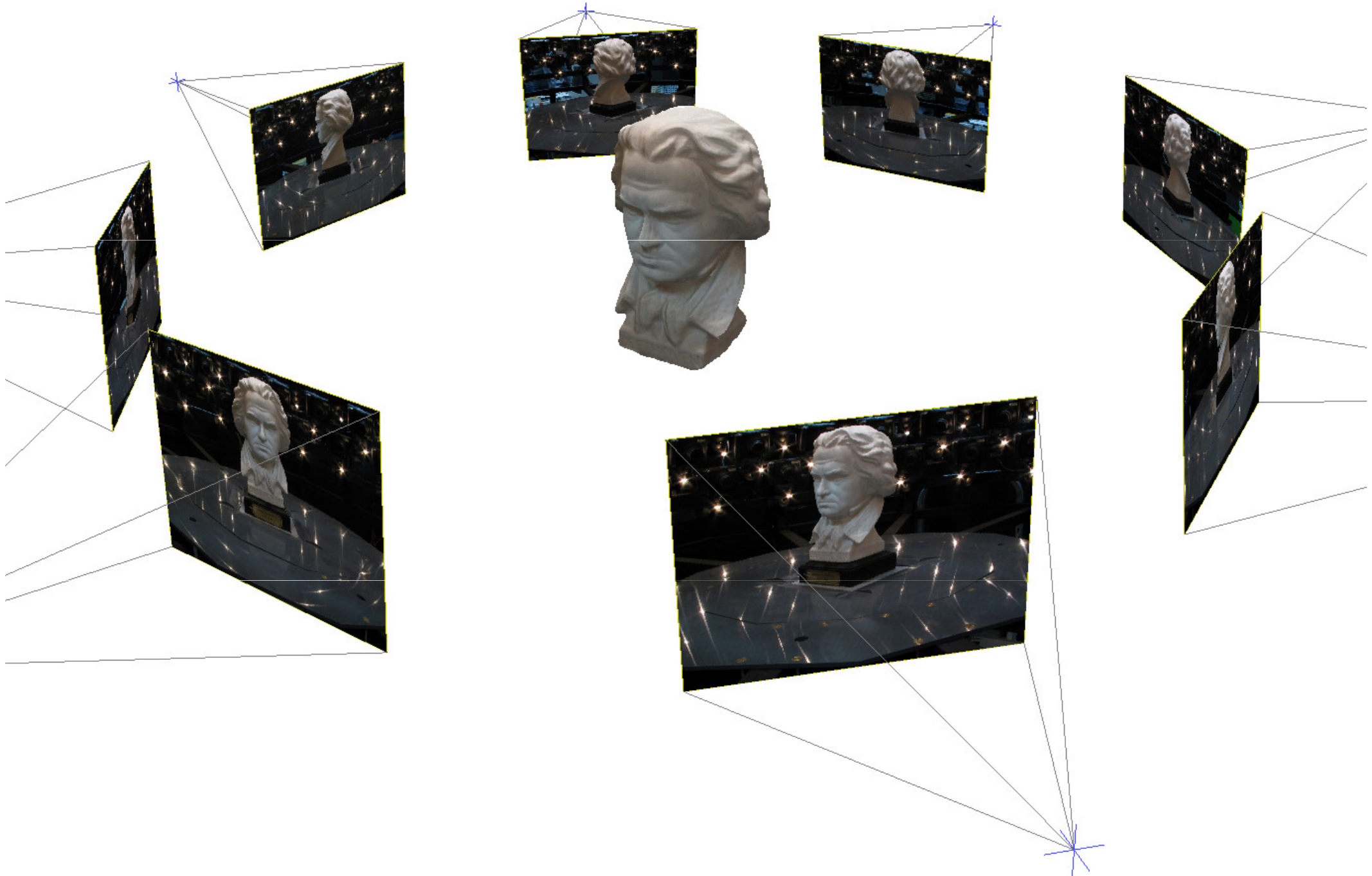
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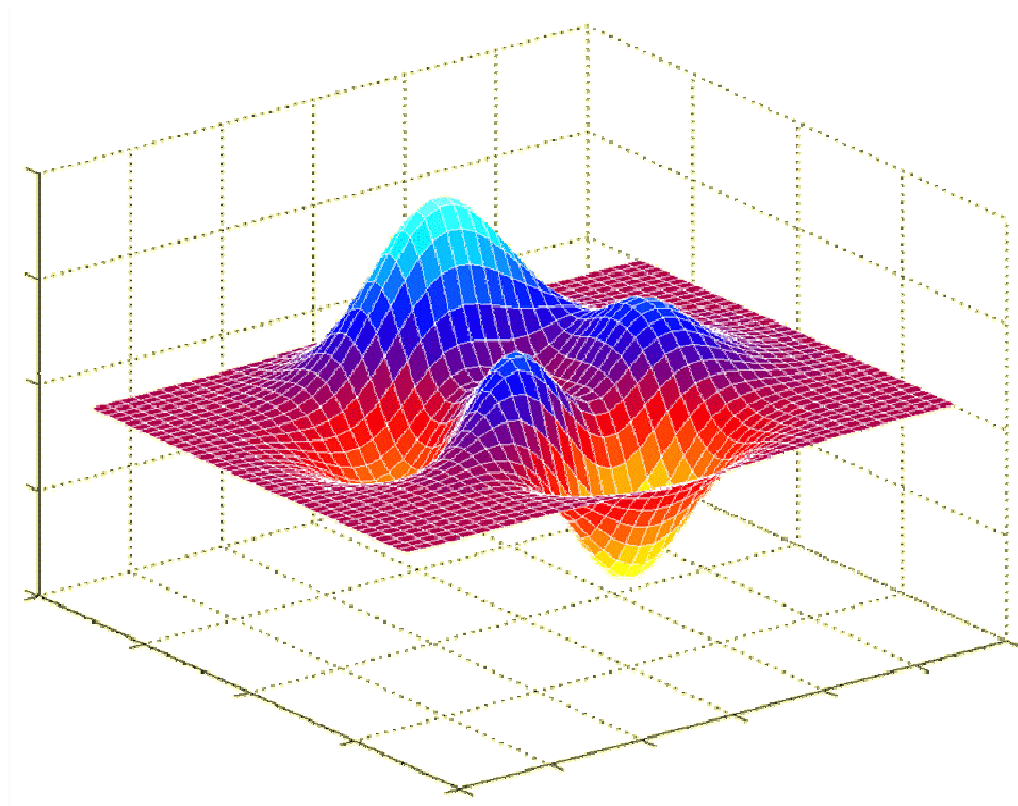


Thomas Pock
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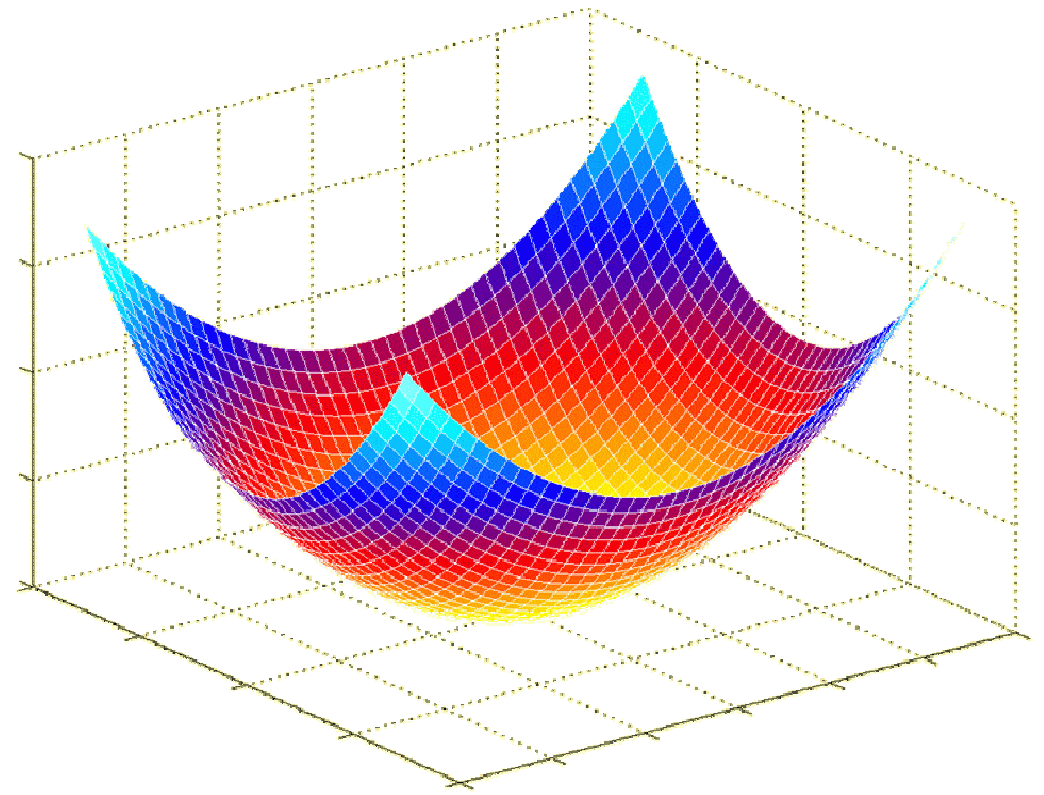


Mutliple view reconstruction





Non-convex energy



Convex energy

Overview



Multiview reconstruction



Super-resolution textures



Stereo & silhouettes



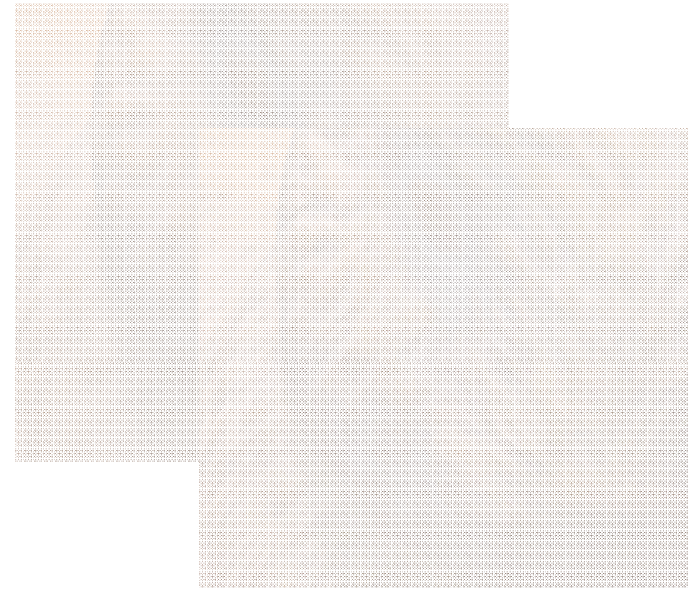
Single view reconstruction

Overview

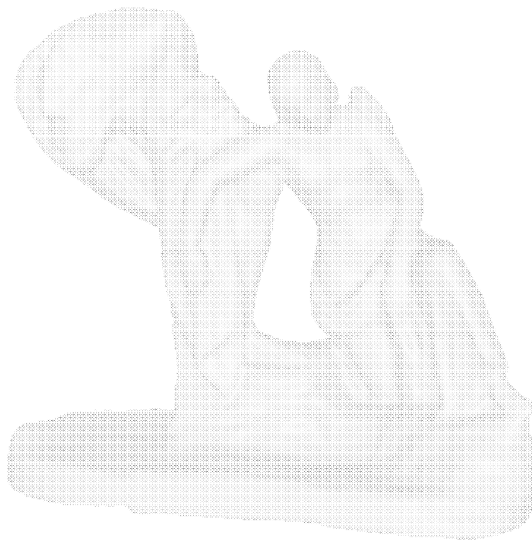
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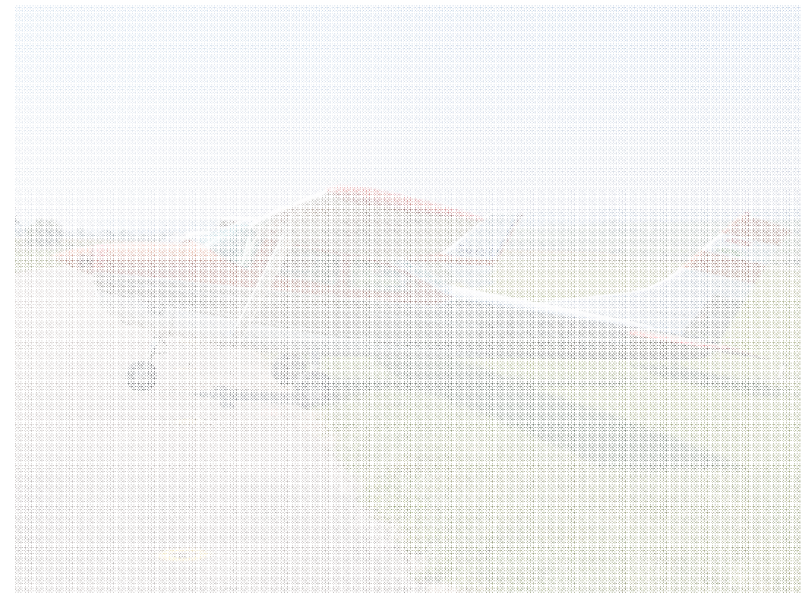
Multiview reconstruction



Super-resolution textures



Stereo & silhouettes



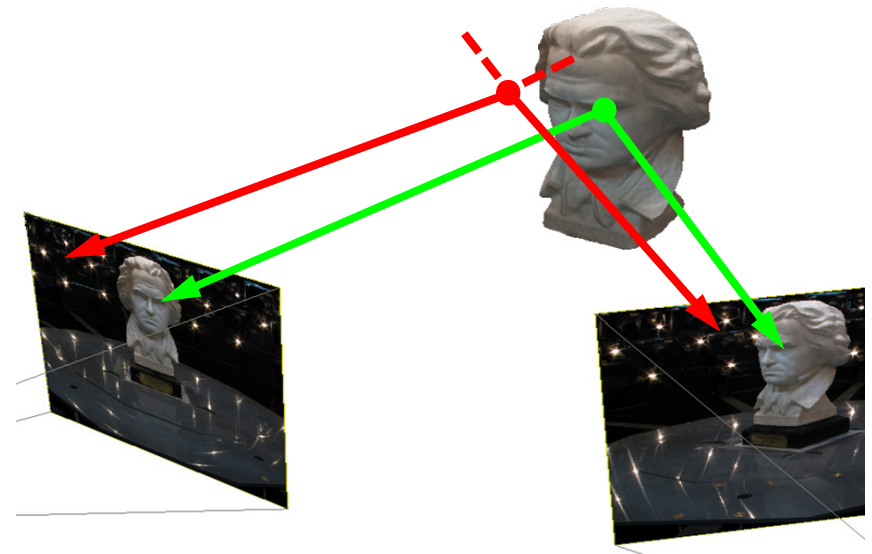
Single view reconstruction

Stereo-weighted minimal surfaces

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$$\rho : (V \subset \mathbb{R}^3) \rightarrow [0, 1]$$

$$E(S) = \int_S \rho(s) ds$$



3D Reconstruction: *Faugeras, Keriven '98, Duan et al. '04*

Segmentation: *Kichenassamy et al. '95, Caselles et al. '95*

Optimal solution is the empty set: $\arg \min_S E(S) = \emptyset$

Resort:

Local optimization: *Faugeras, Keriven TIP '98*

Generative object/background modeling: *Yezzi, Soatto '03,...*

Constrain search space: *Vogiatsis, Torr, Cipolla CVPR '05*

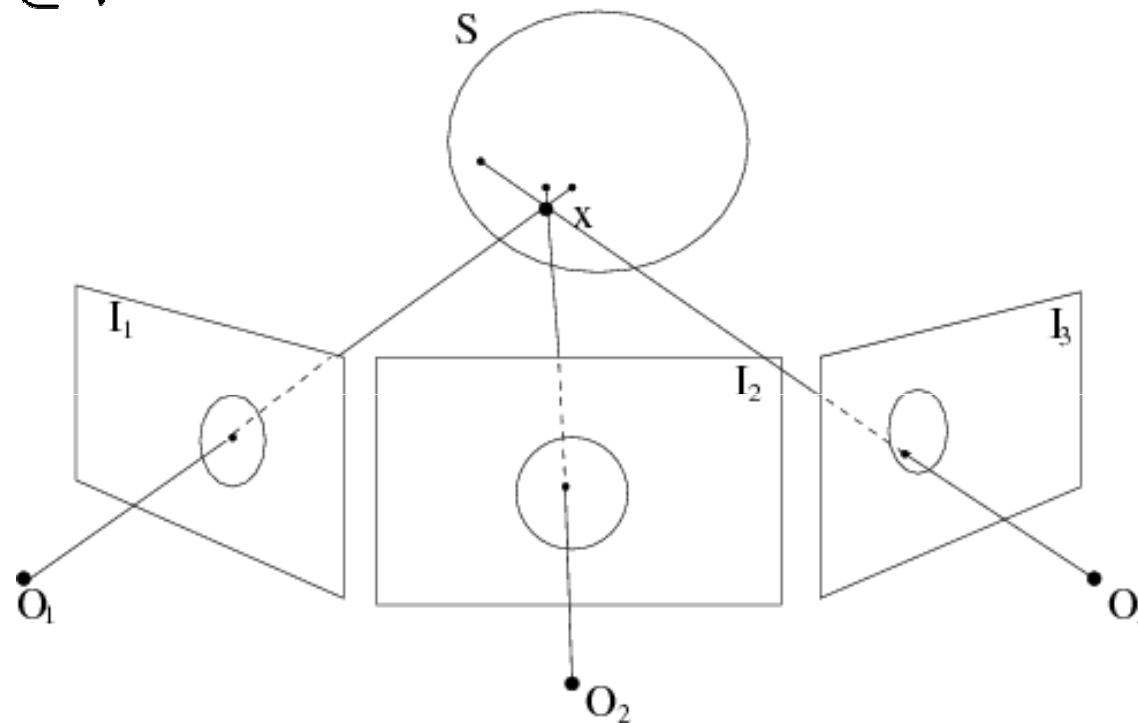
Intelligent ballooning: *Boykov, Lempitsky BMVC '06*

Solution 1: Volumetric photoconsistency

$$E(S) = \int_{S_{in}} \rho_{obj}(x) dx + \int_{S_{out}} \rho_{bck}(x) dx + \int_S \rho(s) ds$$

$$\rho_{obj}, \rho_{bck} : V \rightarrow [0, 1]$$

$$S_{in}, S_{out} \subset V$$



Kolev, Klodt, Brox, Cremers, IJCV '09

Global optima via convex relaxation

$$E(S) = \int_{S_{in}} \rho_{obj} dx + \int_{S_{out}} \rho_{bck} dx + \int_S \rho ds$$



implicit representation

$$E(u) = \int_V \rho_{obj}(x)(1 - u(x)) + \rho_{bck}(x)u(x) dx + \int_V \rho(x)|\nabla u| dx,$$

s. t. $u : V \rightarrow \{0, 1\}$



relaxation

$$E(u) = \int_V \rho_{obj}(x)(1 - u(x)) + \rho_{bck}(x)u(x) dx + \int_V \rho(x)|\nabla u| dx,$$

s. t. $u : V \rightarrow [0, 1]$

Global optima via convex relaxation

$$E(u) = \int_V \rho_{obj} (1 - u(x)) + \rho_{bck} u(x) dx + \int_V \rho |\nabla u| dx, \quad (*)$$

s. t. $u : V \rightarrow [0, 1]$

Theorem: Thresholding a minimizer u^* of the relaxed problem $(*)$ leads to an optimal solution of the original binary problem:

$$u_{opt}(x) = 1_{u^* \geq \mu}(x) = \begin{cases} 1, & \text{if } u^*(x) \geq \mu \\ 0, & \text{if } u^*(x) < \mu \end{cases}$$

for any threshold $\mu \in (0, 1)$.

Chan, Esedoglu, Nikolova, TIP '06

Kolev, Klodt, Brox, Esedoglu, Cremers, EMMCVPR '07, IJCV '09

A thresholding theorem

Let

$$u^* : \Omega \rightarrow [0, 1]$$

be a (real-valued) minimizer of

$$E(u) = \int_{\Omega} f u + |\nabla u| dx.$$

Then for any threshold $\mu \in (0, 1)$, the binary function

$$1_{u^* \geq \mu}(x)$$

is a global minimizer of the original binary problem.

A thresholding theorem

Proof: 1) $u(x) = \int_0^1 1_{u \geq \mu}(x) d\mu$ (layer cake formula)

2) $\int_{\Omega} |\nabla u| dx = \int_0^1 \int_{\Omega} |\nabla 1_{u \geq \mu}(x)| dx d\mu$ (coarea formula)

$$E(u) = \int_{\Omega} f u + |\nabla u| dx = \int_0^1 \int_{\Omega} f \mathbf{1}_{u \geq \mu} + |\nabla \mathbf{1}_{u \geq \mu}| dx d\mu = \int_0^1 E(\mathbf{1}_{u \geq \mu}) d\mu$$

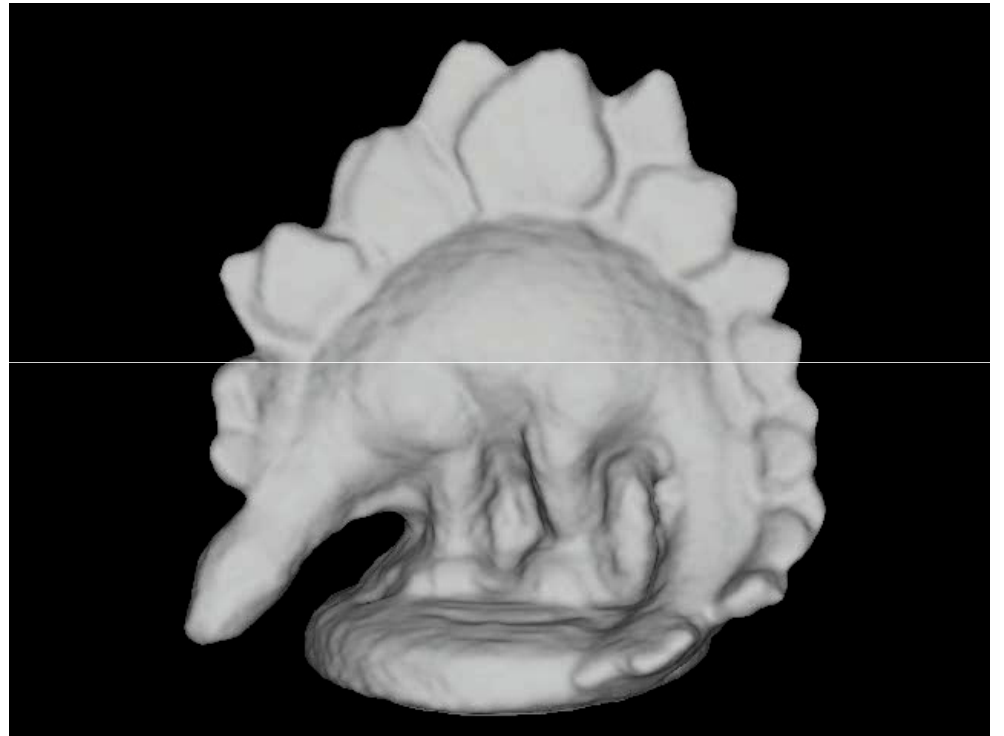
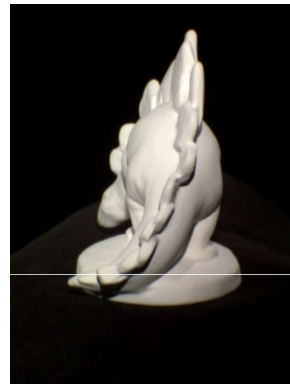
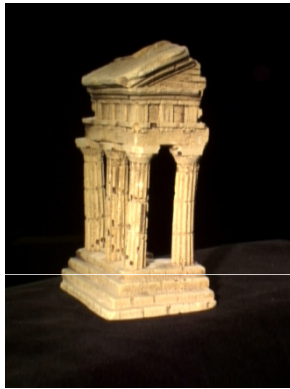
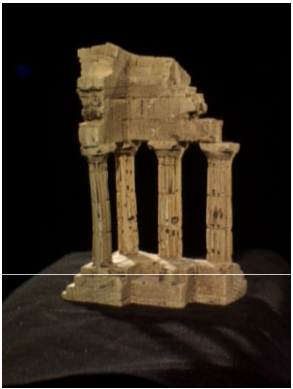
If $1_{u^* \geq \mu}$ is not minimizer, i.e. there exists a set $\Sigma \subset \Omega$

$$E(1_{\Sigma}) < E(1_{u^* \geq \mu})$$

$$\Rightarrow E(1_{\Sigma}) = \int_0^1 E(1_{\Sigma}) d\mu < \int_0^1 E(1_{u^* \geq \mu}) d\mu = E(u^*) \quad (\text{i.e. } u^* \text{ not minimizer})$$

Reconstruction results

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Discrete versus continuous optimization

Level sets	Convex formulation	Graph cuts
Continuous Local optima	Continuous Global optima*	Discrete Global optima*

* for certain functionals only

Klodt, Schoenmann, Kolev, Schikora, Cremers, ECCV '08

Discrete versus continuous optimization

Level sets	Convex formulation	Graph cuts
Continuous Local optima	Continuous Global optima* Parallel implementations	Discrete Global optima*

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Klodt, Schoenmann, Kolev, Schikora, Cremers, ECCV '08

Discrete versus continuous optimization

Level sets	Convex formulation	Graph cuts
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Klodt, Schoenmann, Kolev, Schikora, Cremers, ECCV '08

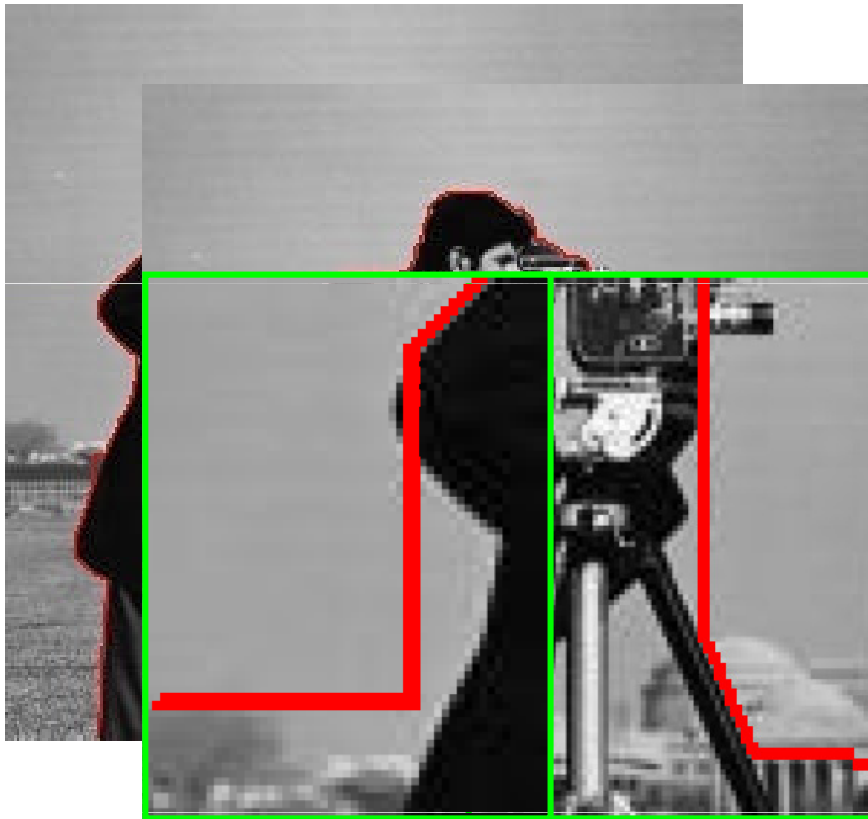
Discrete versus continuous optimization

Level sets	Convex formulation	Graph cuts
Continuous Local optima	Continuous Global optima* Parallel implementations	Discrete Global optima* Memory limitations Metrication errors

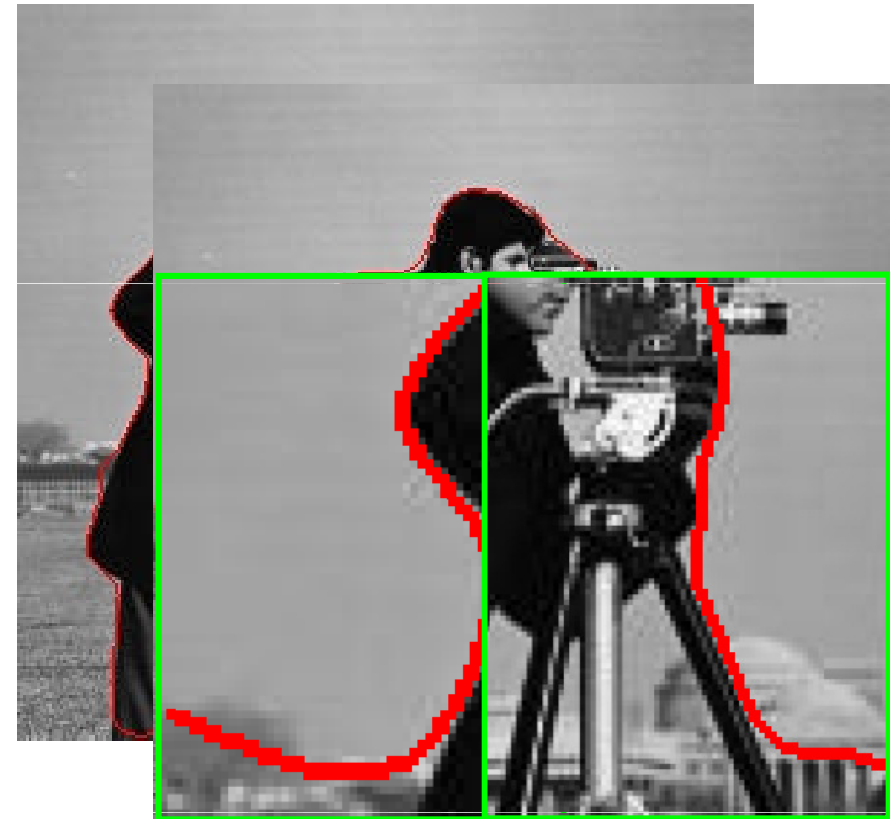
* for certain functionals only

Klodt, Schoenmann, Kolev, Schikora, Cremers, ECCV '08

Metrication errors and consistency

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Discrete graph cut optimization
(4-connected grid)



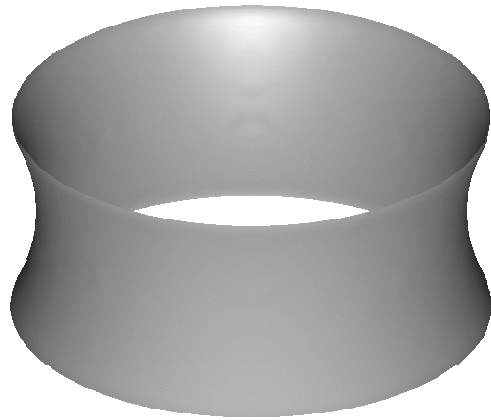
Continuous convex formulation
(4-connected grid)

Improvements:

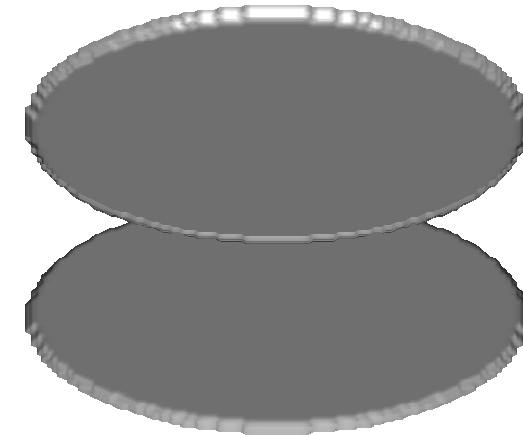
Larger neighborhoods (*Boykov, Kolmogorov '03, Kirsanov, Gortler '04*)

Continuous Maximum Flow (*Appleton, Talbot '06*)

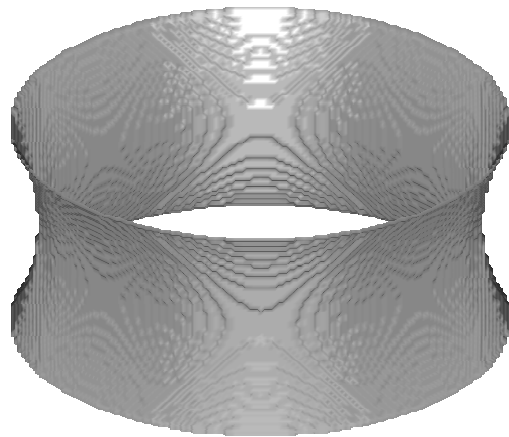
A minimal surface example: The catenoid



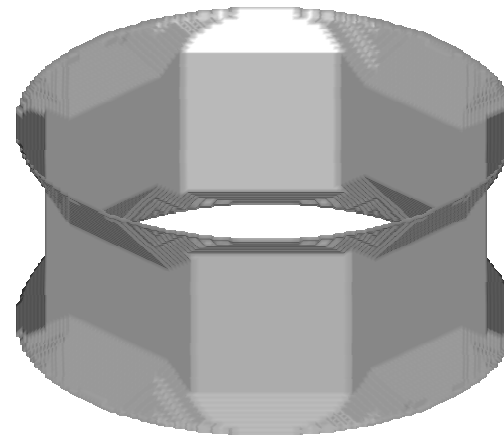
True solution



Graph cut
(6-connected grid)



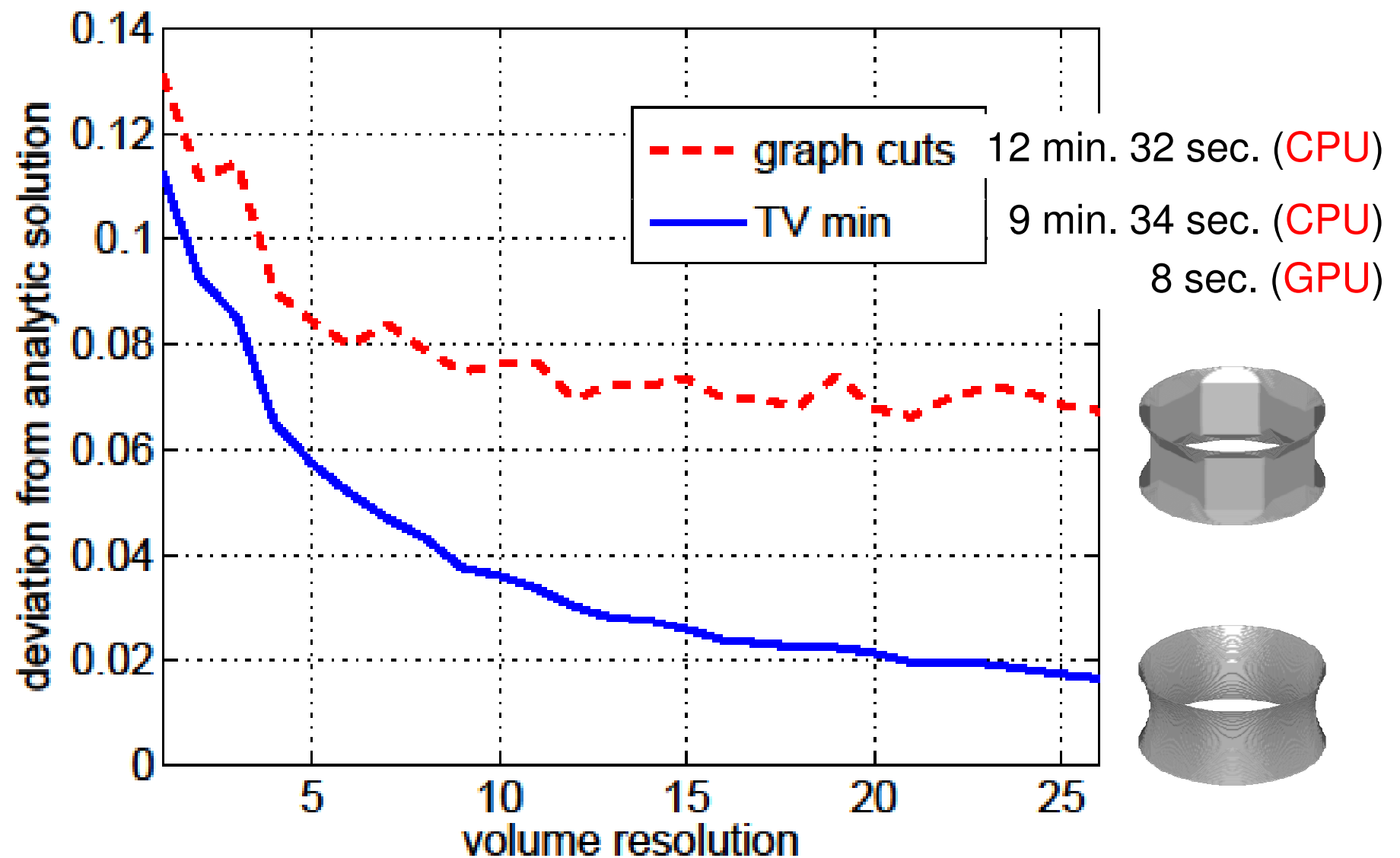
Convex formulation
(6-connected grid)



Graph cut
(26-connected grid)

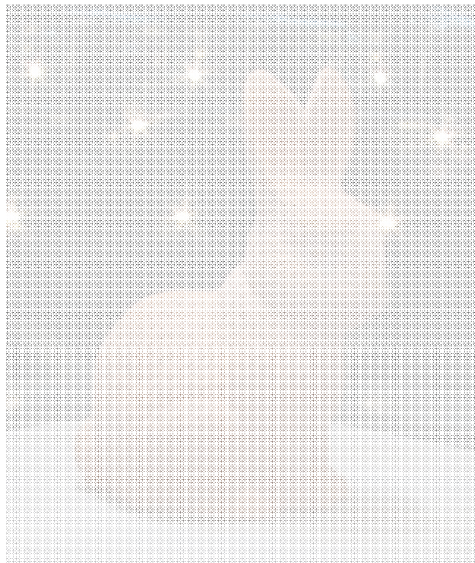
Klodt, Schoenmann, Kolev, Schikora, Cremers, ECCV '08

Metrication errors and consistency

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Klodt, Schoenmann, Kolev, Schikora, Cremers, ECCV '08

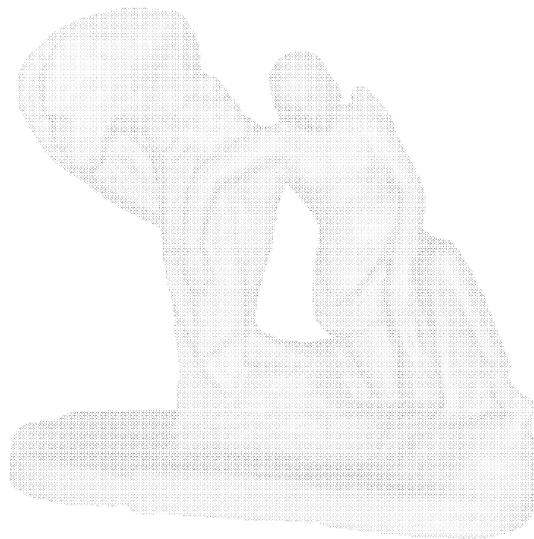
Overview



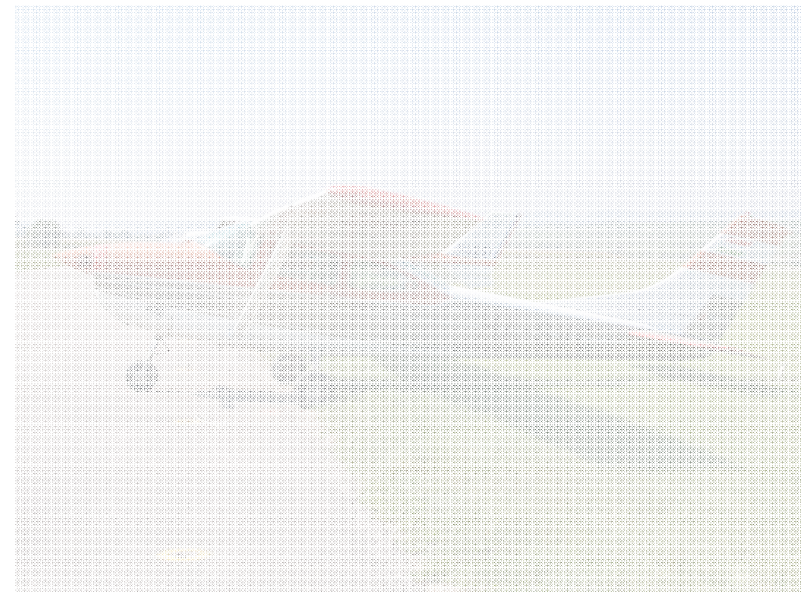
Multiview reconstruction



Super-resolution textures

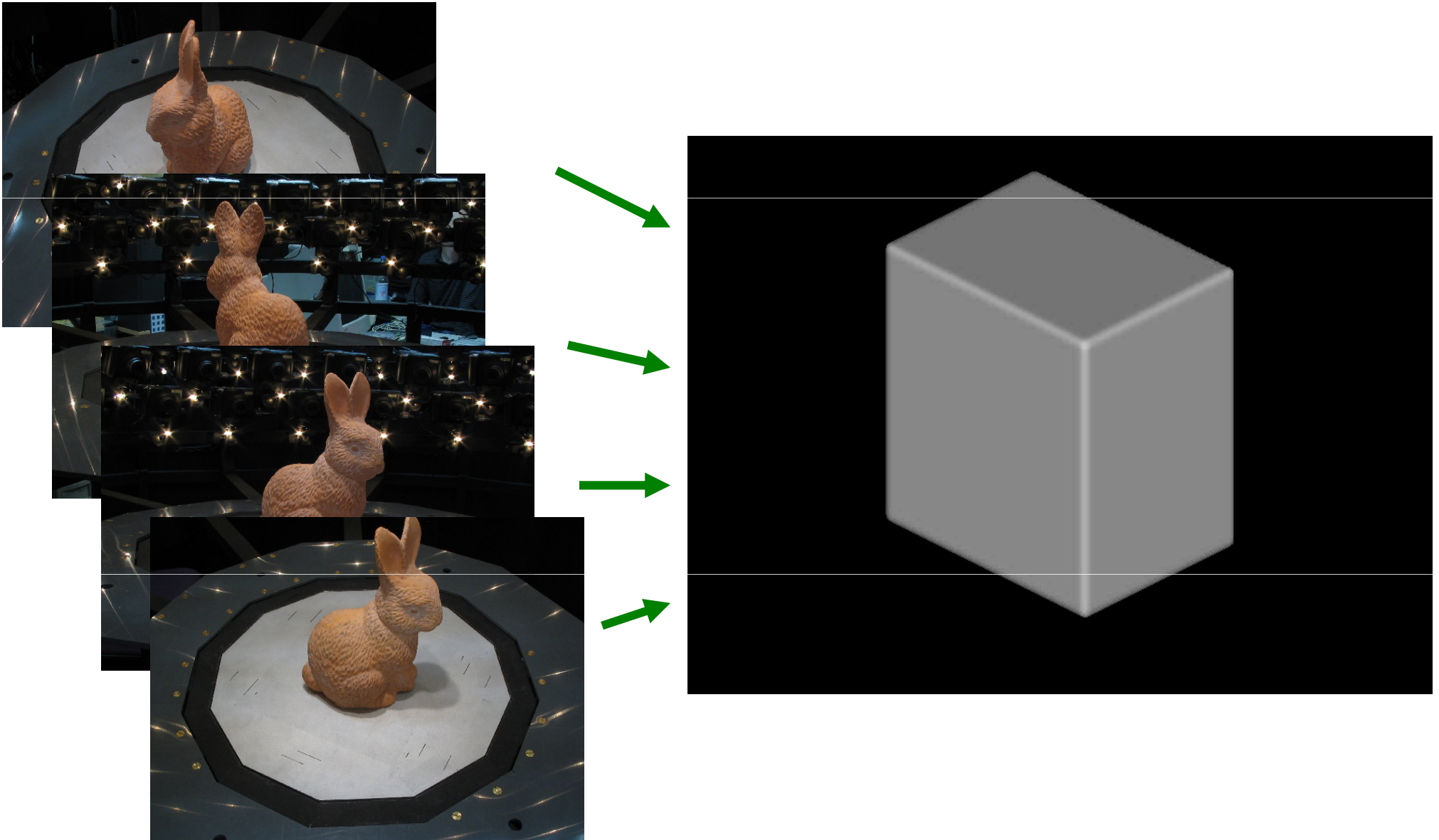


Stereo & silhouettes



Single view reconstruction

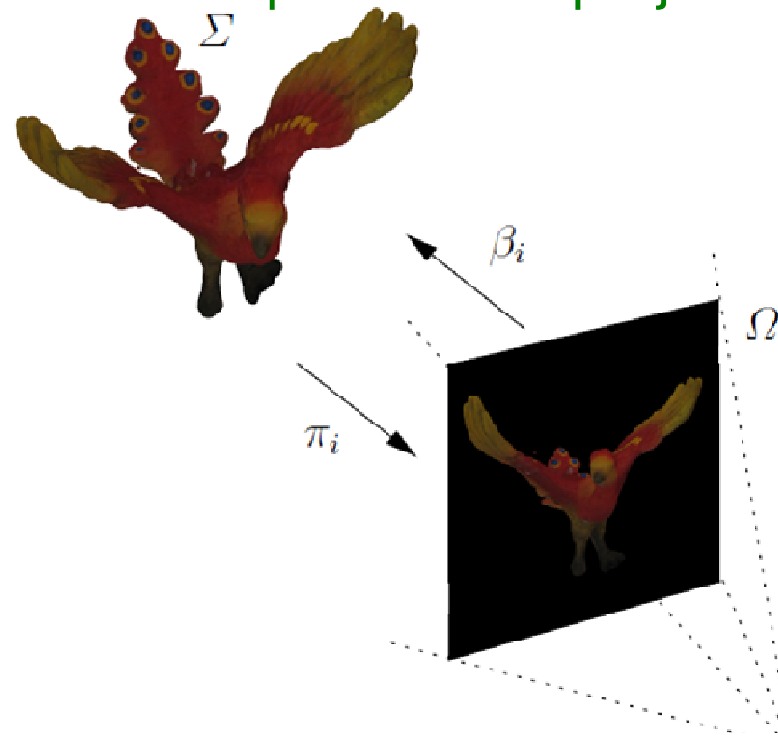
Evolution to global optimum

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Super-resolution texture estimation

Given all images $\mathcal{I}_i : \Omega_i \rightarrow \mathbb{R}^3$, determine the surface color $T : S \rightarrow \mathbb{R}^3$

$$\min_T \sum_{i=1}^n \int_{\Omega_i} \left(\underbrace{b *}_{\text{blur \& downsample}} (T \circ \underbrace{\beta_i}_{\text{back-projection}}) - \mathcal{I}_i \right)^2 dx + \lambda \int_S \|\nabla_S T\| ds$$



Goldlücke, Cremers, ICCV '09, DAGM '09

Super-resolution texture estimation

Given all images $\mathcal{I}_i : \Omega_i \rightarrow \mathbb{R}^3$, determine the surface color $T : S \rightarrow \mathbb{R}^3$

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Euler-Lagrange equation gives a PDE on the surface:

$$\frac{dE}{dT} = -\operatorname{div}_S \left(\frac{\nabla_S T}{\|\nabla_S T\|_S} \right) + \sum_{i=1}^n \frac{v_i}{\lambda} \left((\mathcal{J}_i \mathcal{D}_i) \circ \pi_i \right) = 0$$

where $\mathcal{D}_i = \bar{b} * (b * (T \circ \beta_i) - \mathcal{I})$ and $\mathcal{J}_i = \left\| \frac{\partial \beta_i}{\partial x} \times \frac{\partial \beta_i}{\partial y} \right\|^{-1}$.

Conformal parameterization of the surface \longrightarrow PDE on charts.

Goldlücke, Cremers, ICCV '09, DAGM '09

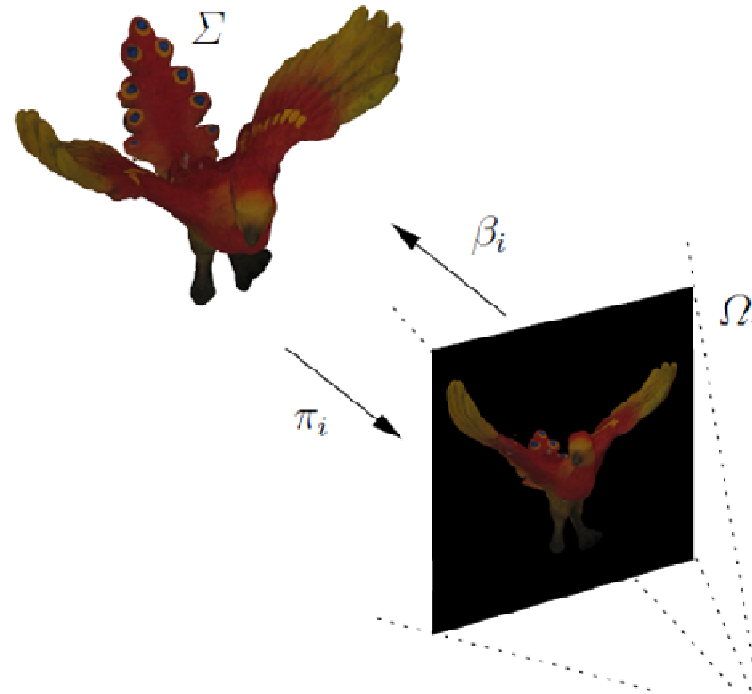
Joint estimation of geometry & super-resolution texture

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Determine normal displacement $D: S \rightarrow \mathbb{R}$ and surface color $T: S \rightarrow \mathbb{R}^3$

$$\min_{D,T} \sum_{i=1}^n \int_{\Omega_i} \left(b * (T \circ \beta_i^D) - \mathcal{I}_i \right)^2 dx + \lambda \int_S \|\nabla_S T\| ds$$

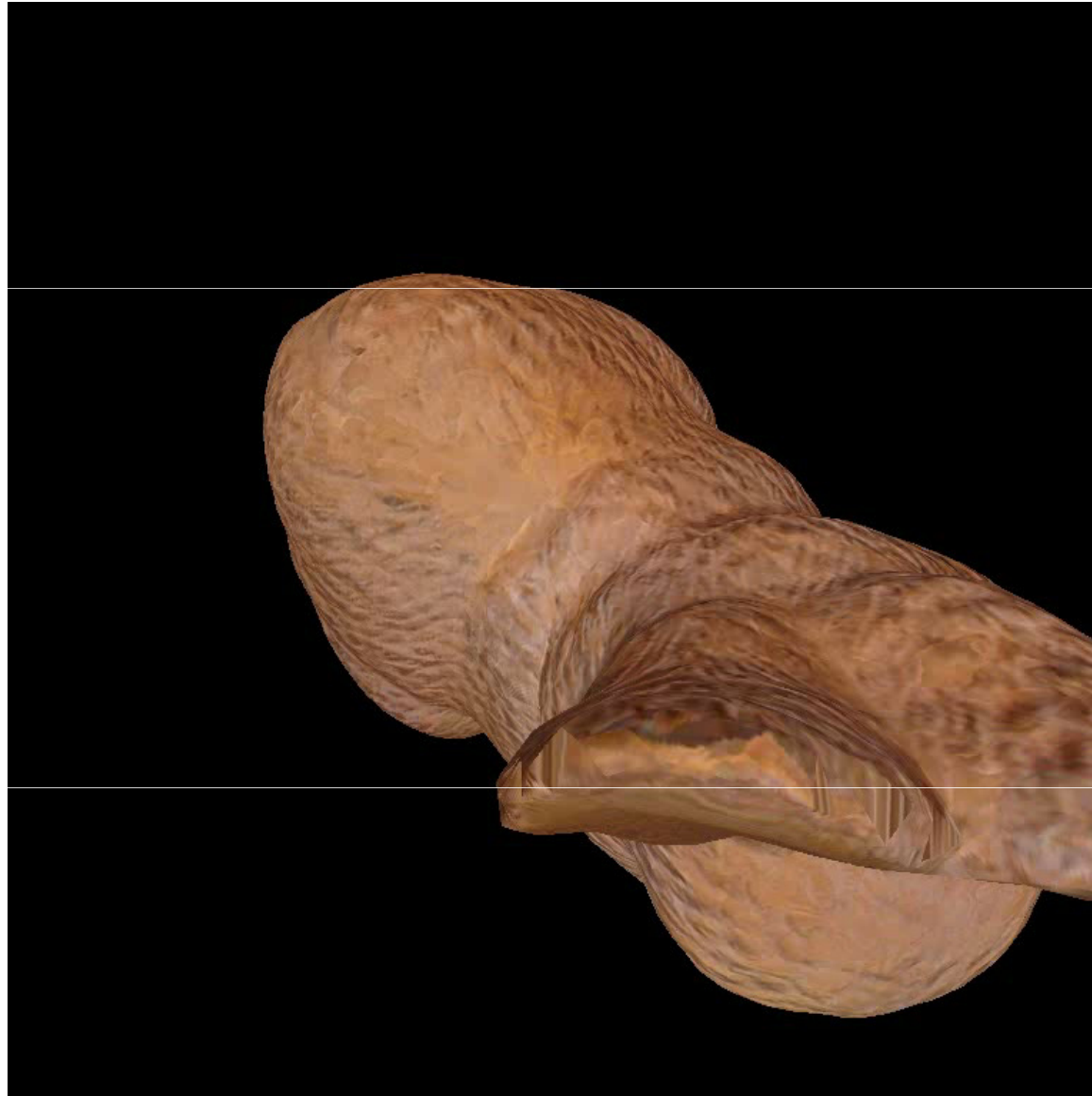
geometry-dependent back-projection



Goldlücke, Cremers, ICCV '09, DAGM '09

Super-resolution texture estimation

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Goldlücke, Cremers, ICCV '09, DAGM '09

Evolution of the super-resolution process

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Goldlücke, Cremers, ICCV '09, DAGM '09

Super-resolution texture estimation

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Weighted average



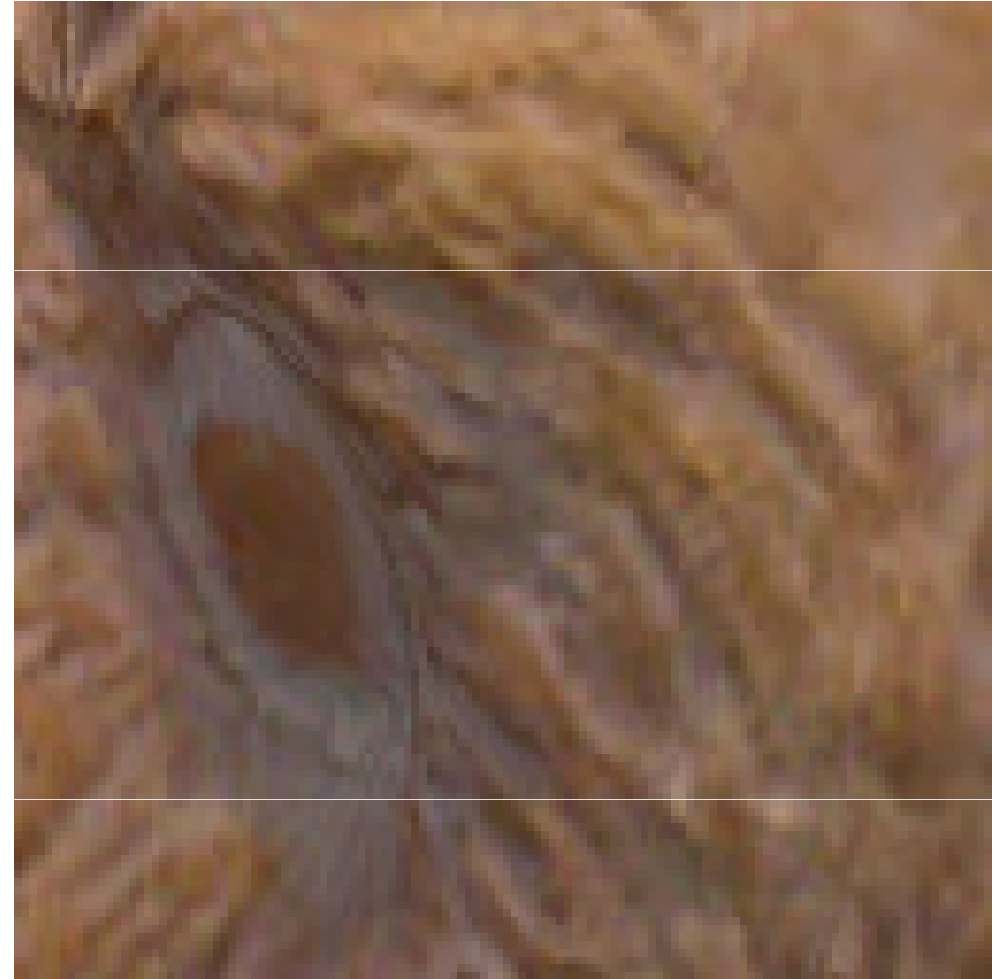
Super-resolution texture

Goldlücke, Cremers, ICCV '09, DAGM '09

Super-resolution texture estimation

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Input image



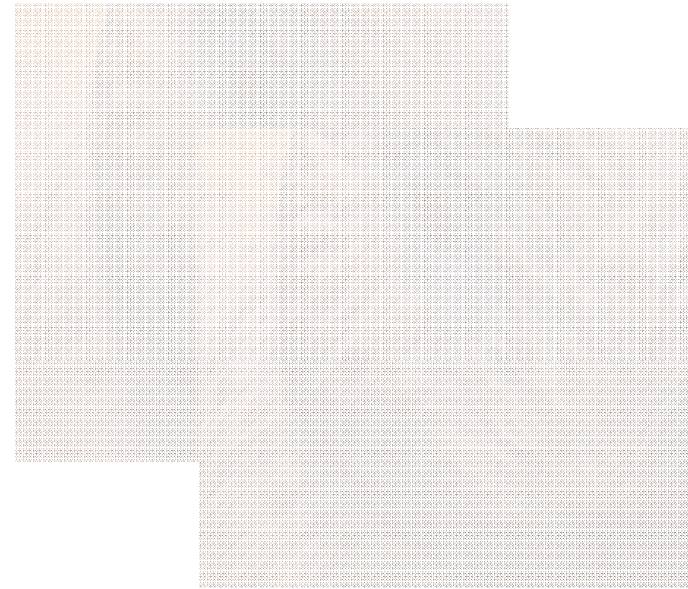
Super-resolution texture

Goldlücke, Cremers, ICCV '09, DAGM '09

Overview



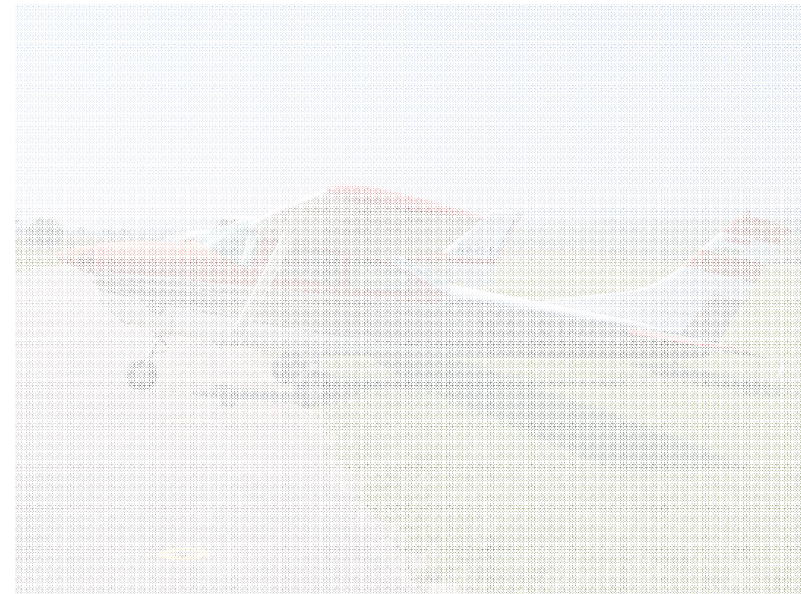
Multiview reconstruction



Super-resolution textures



Stereo & silhouettes



Single view reconstruction

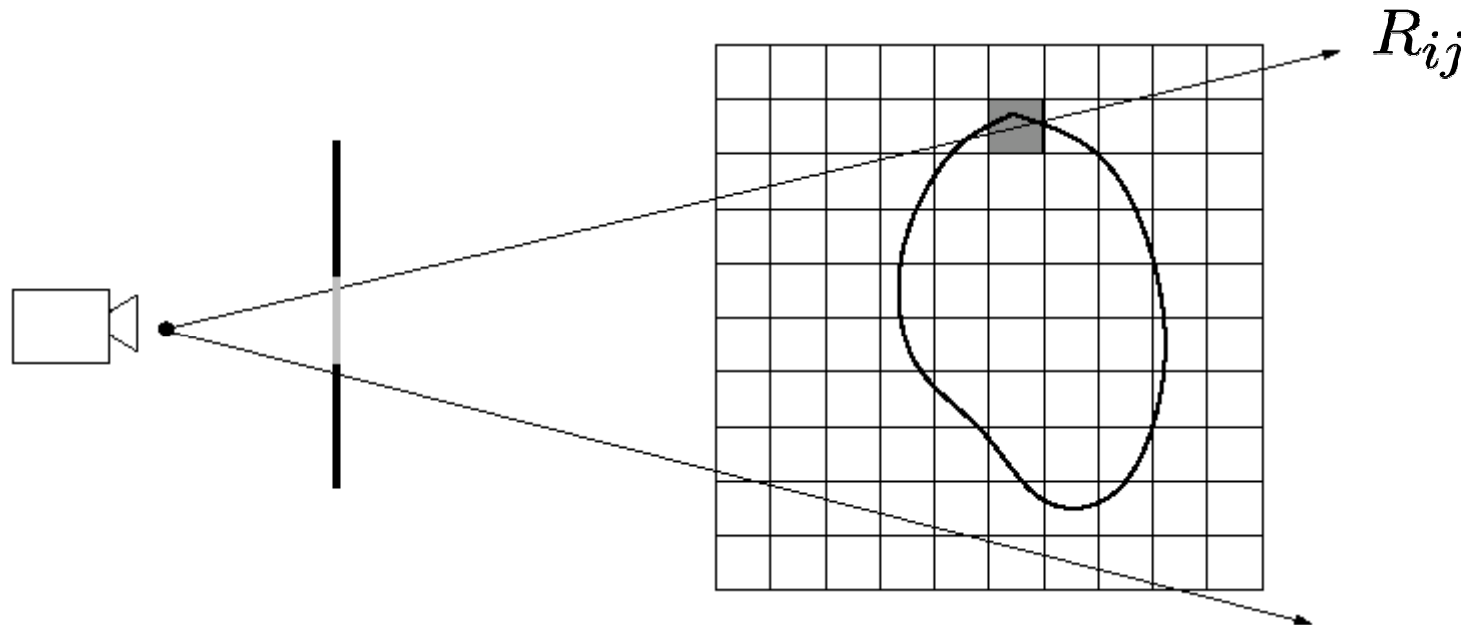
Solution 2: Imposing silhouette consistency

$$\min_S \int_S \rho \, dS$$

$$\text{s. t. } \pi_i(S) = S_i \quad \forall i = 1, \dots, n$$

$$\pi_i : V \rightarrow \Omega_i$$

$$S_i \subset \Omega_i$$



Kolev, Cremers, ECCV '08, PAMI 2010

Silhouette consistency as a convex constraint

$$E(S) = \int_S \rho(x) dS,$$

$$\text{s. t.} \quad \pi_i(S) = S_i \quad \forall i = 1, \dots, n$$



implicit representation & relaxation

$$\Sigma = \left\{ \begin{array}{l} E(u) = \int_V \rho(x) |\nabla u(x)| dx \\ \text{s. t.} \quad \cancel{u : V \rightarrow \{0, 1\}} \quad u : V \rightarrow [0, 1] \\ \int_{R_{ij}} u(x) dx \geq \delta \quad \text{if } j \in S_i \\ \int_{R_{ij}} u(x) dx = 0 \quad \text{if } j \notin S_i \end{array} \right.$$

Proposition: The set Σ of silhouette-consistent solutions is convex.

Kolev, Cremers, ECCV '08, PAMI 2010

Silhouette consistency as a convex constraint

$$E(u) = \int_V \rho(x) |\nabla u(x)| dx$$

$$\text{s. t.} \quad u : V \rightarrow [0, 1]$$

$$\int_{R_{ij}} u(x) dx \geq 1 \quad \text{if } j \in S_i$$

$$\int_{R_{ij}} u(x) dx = 0 \quad \text{if } j \notin S_i$$



thresholding

$$R_{obj}^S = \{x \in V \mid u(x) > \mu\}$$

$$R_{bck}^S = \{x \in V \mid u(x) < \mu\}, \text{ where}$$

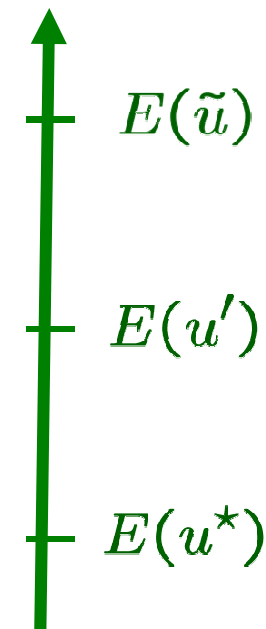
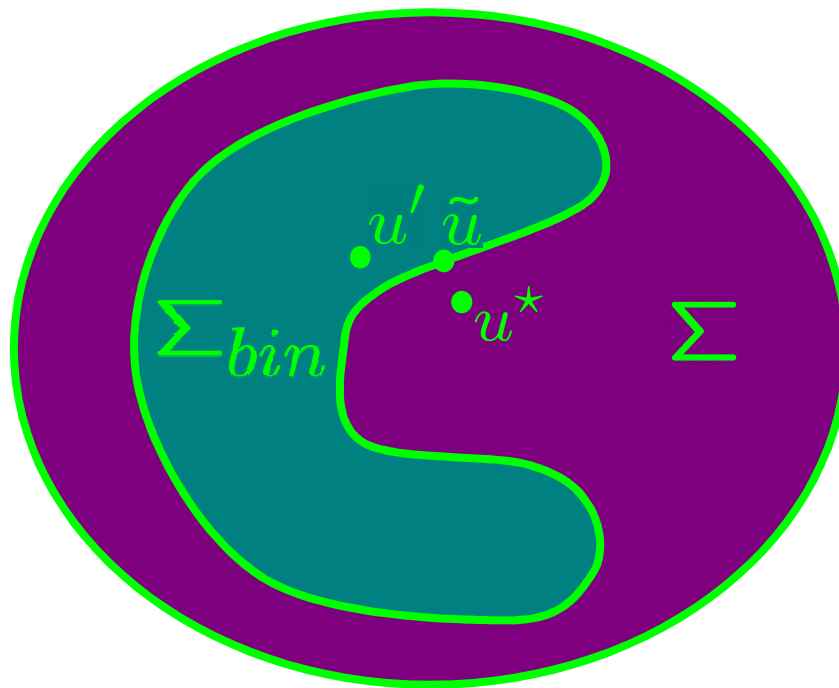
$$\mu = \min \left\{ \left(\min_{i \in \{1, \dots, n\}, j \in S_i} \max_{x \in R_{ij}} u^*(x) \right), 0.5 \right\}$$

Kolev, Cremers, ECCV '08, PAMI 2010

Bounded optimality

$$u^* = \arg \min_{u \in \Sigma} E(u)$$

$$u' = \arg \min_{u \in \Sigma_{bin}} E(u)$$



$$E(\tilde{u}) - E(u') \leq E(\tilde{u}) - E(u^*)$$

Kolev, Cremers, ECCV '08, PAMI 2010

Numerical optimization via lagged diffusivity

Euler-Lagrange equation

$$\operatorname{div} \left(\rho \frac{\nabla u}{|\nabla u|} \right) = 0$$

linearization \downarrow $g := \frac{\rho}{|\nabla u|}$

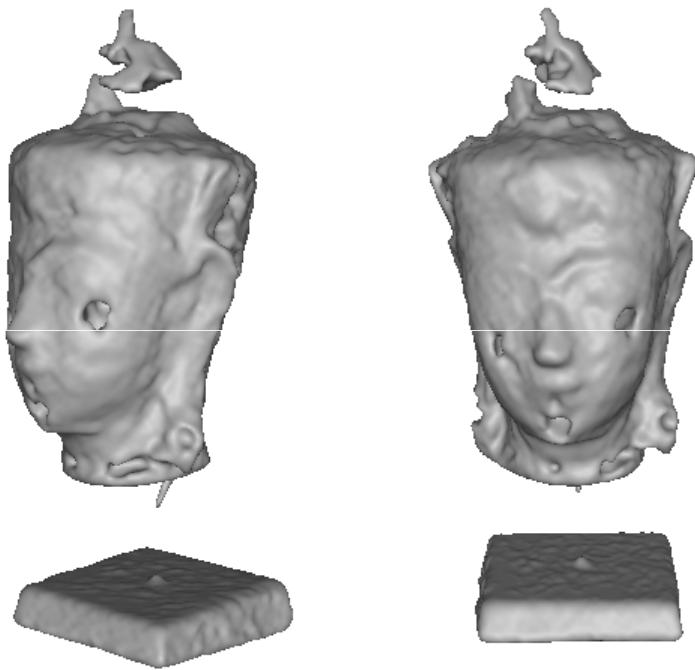
$$\operatorname{div} (g \nabla u) = 0$$

discretization \downarrow

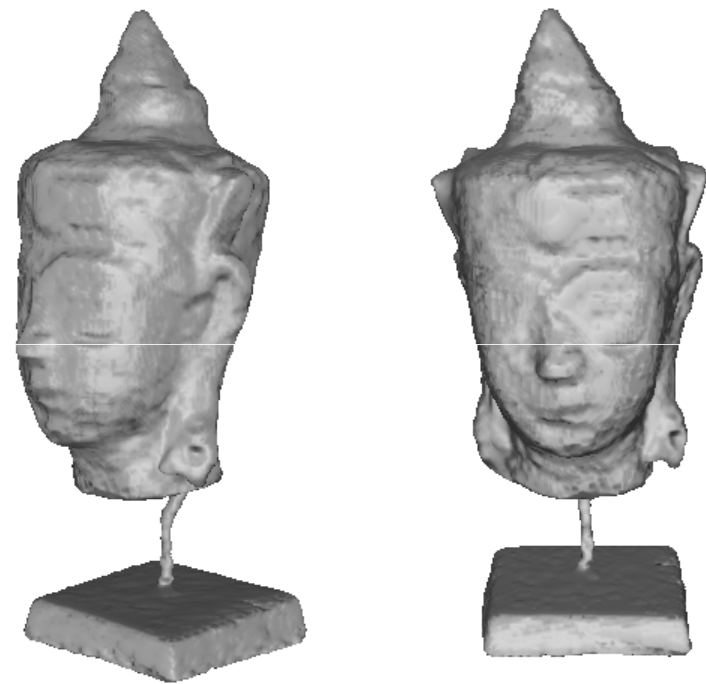
$$\begin{pmatrix} \star & \star & & & \star & & \\ \star & \star & \star & & & \star & \\ & \star & \star & \star & & & \star \\ & & \star & \star & \star & & \\ & & & \star & \star & \star & \\ \star & & & & \star & \star & \star \\ & \star & & & \star & \star & \star \\ & & \star & & & \star & \star \end{pmatrix} \begin{pmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

sparse system of linear equations, solved by Successive Overrelaxation

Reconstruction of non-Lambertian objects

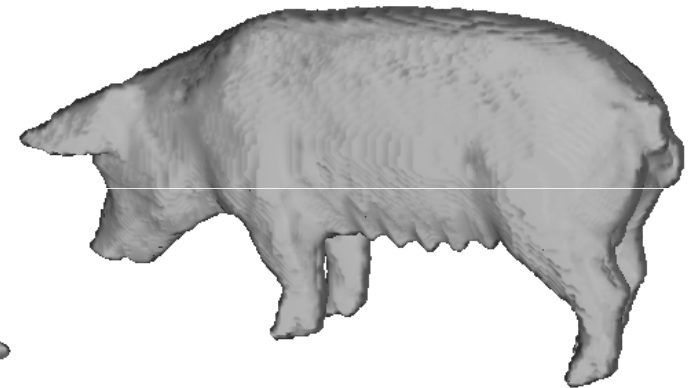
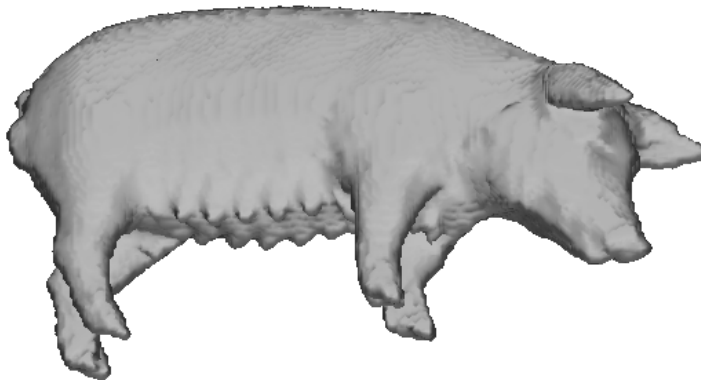
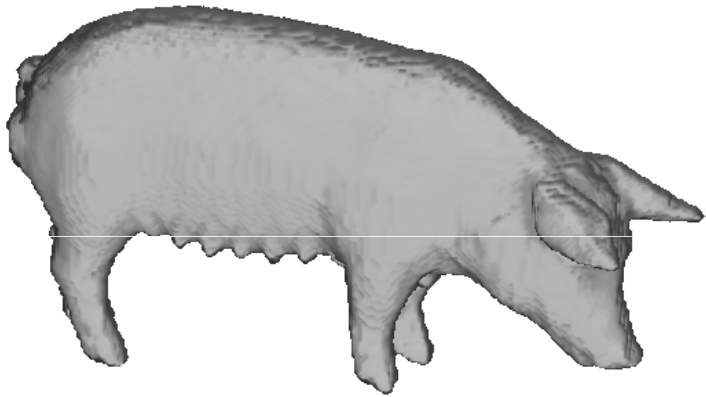
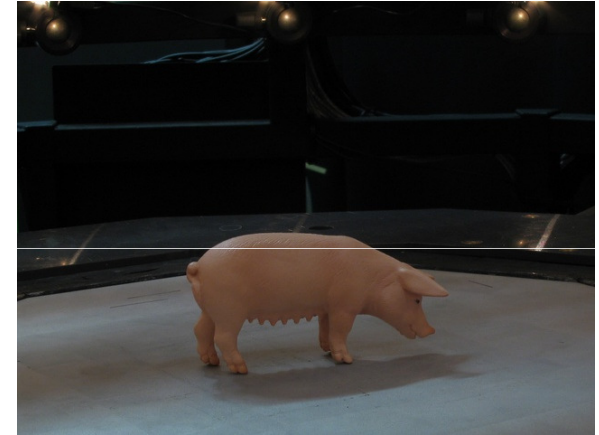
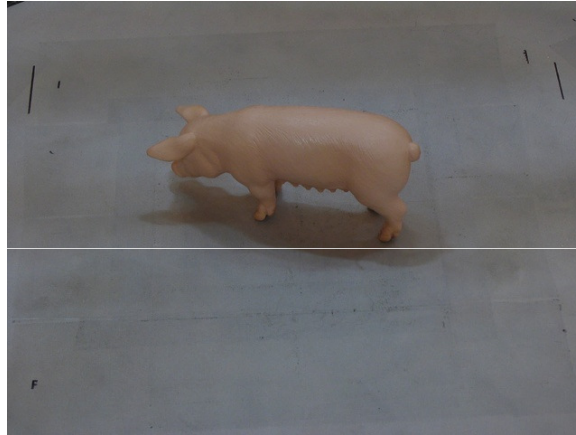
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propagation scheme



silhouette constraints

Reconstruction of low-textured objects

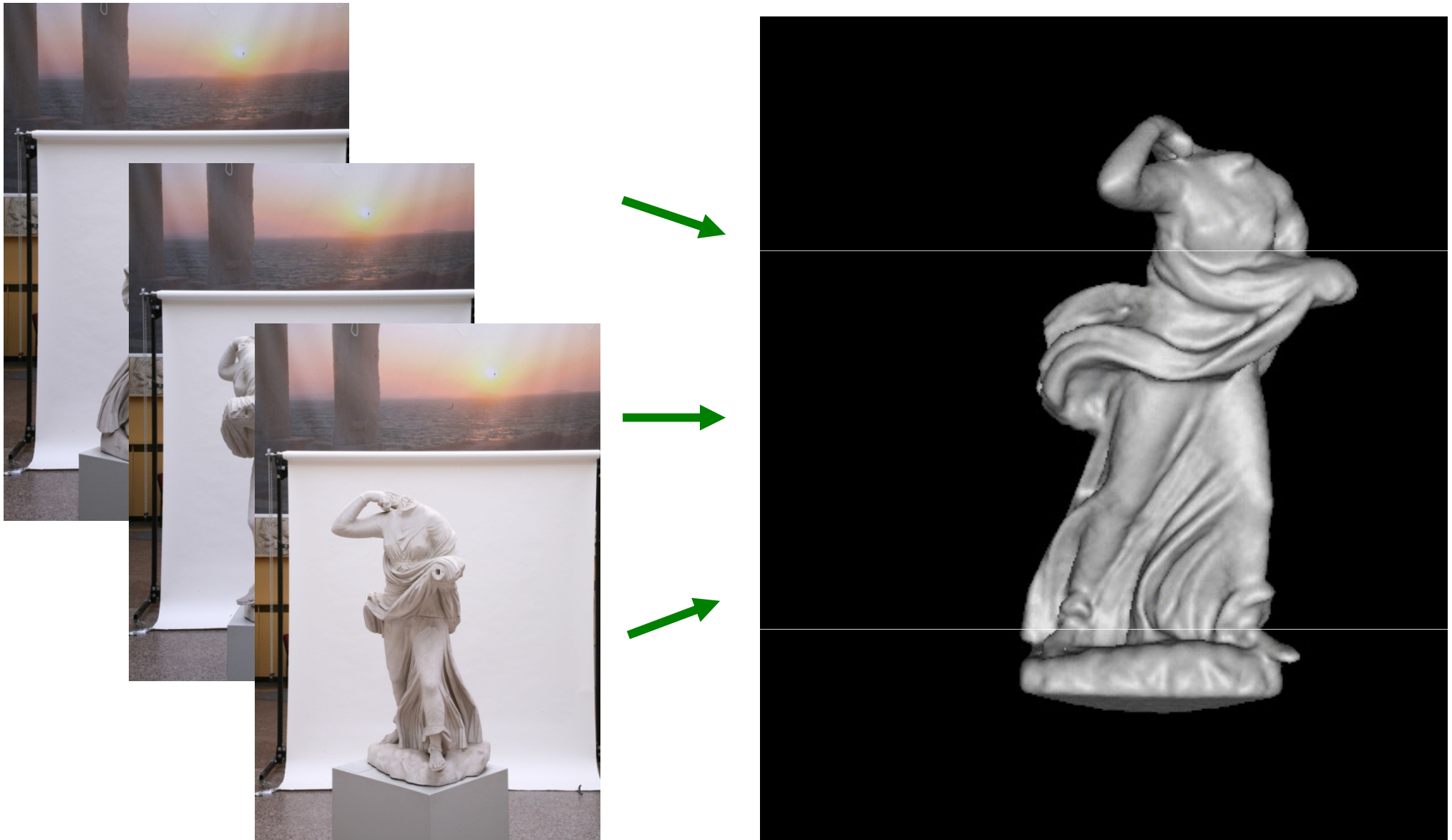
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Reconstruction of fine-scale structures



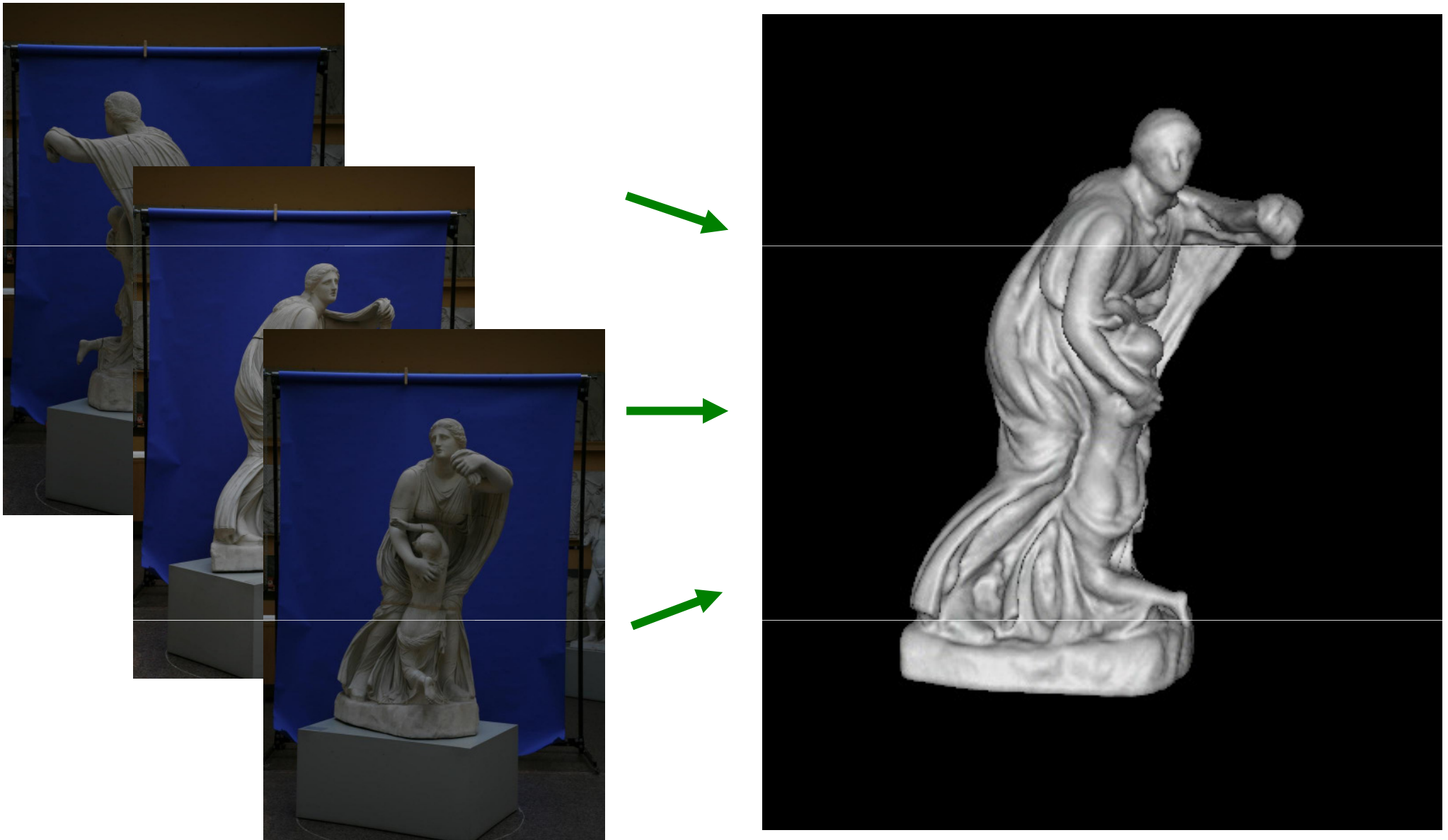
Image data courtesy of Yasutaka Furukawa.

Reconstructing the Niobids statues (450 B.C.)

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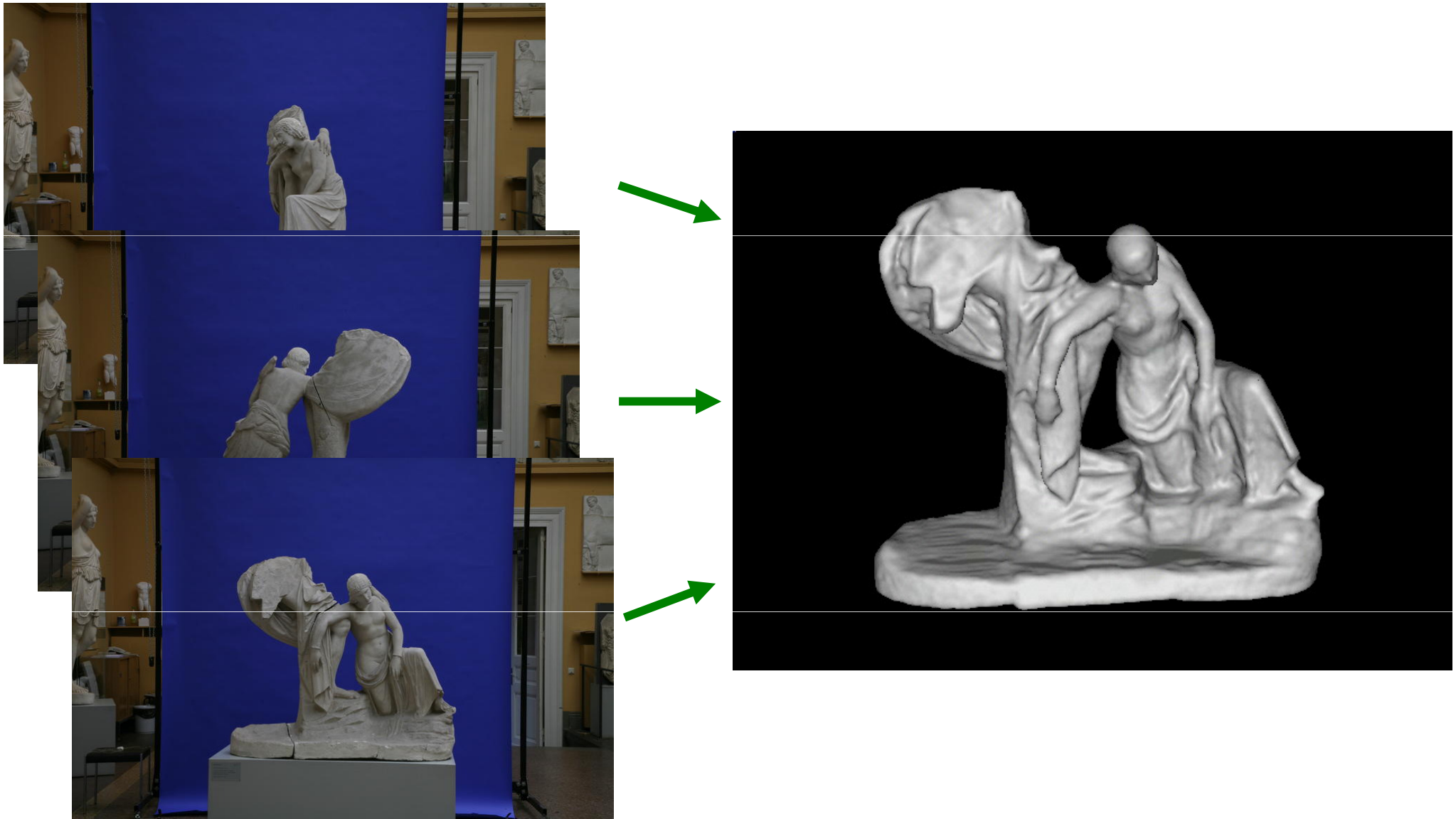
Kolev, Cremers, ECCV '08, PAMI 2010

Reconstructing the Niobids statues (450 B.C.)

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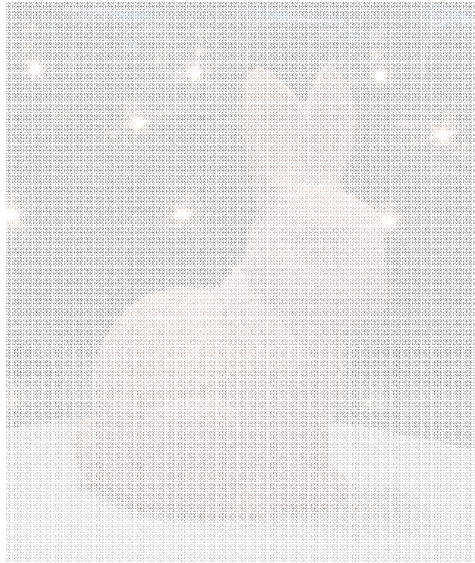
Kolev, Cremers, ECCV '08, PAMI 2010

Reconstructing the Niobids statues (450 B.C.)

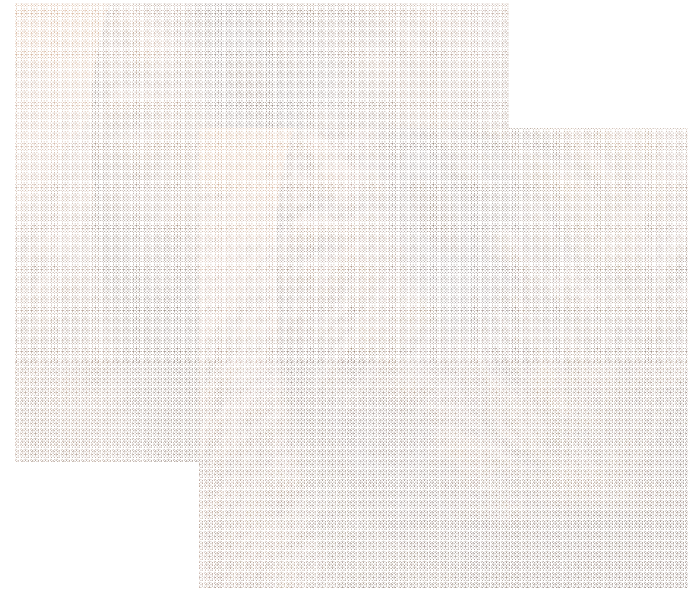
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Kolev, Cremers, ECCV '08, PAMI 2010

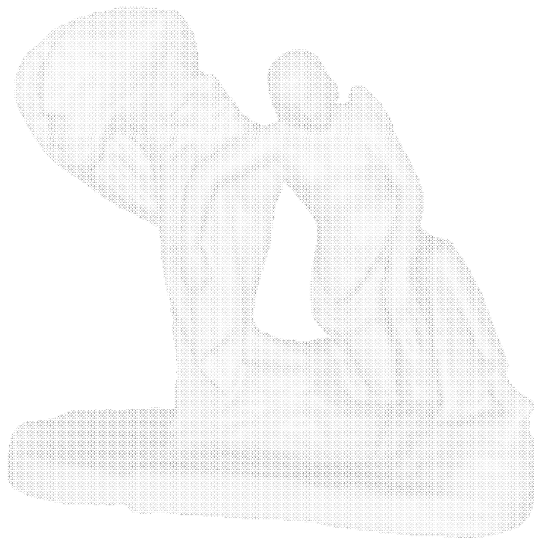
Overview



Multiview reconstruction



Super-resolution textures



Stereo & silhouettes



Single view reconstruction

Variational single view reconstruction

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Can we recover geometry from a single image?

Yes: Shape-from-shading, shape-from-focus, shape from symmetry,...

Problem: Most approaches do not work well for real-world images.

Variational single view reconstruction

Silhouette-based approaches:

Horry et al. Siggraph '97, Criminisi et al. IJCV '00,

Hoiem et al. Siggraph '05, Prasad et al. CVPR '06,...

Goal: Simple variational approach with minimal user interaction.

Solution: Fixed-volume silhouette-consistent minimal surface.

$$\min_u \int_{\mathbb{R}^3} |\nabla u| \, dx, \quad \text{s.t.} \quad \int_{\mathbb{R}^3} u \, dx = V_0$$

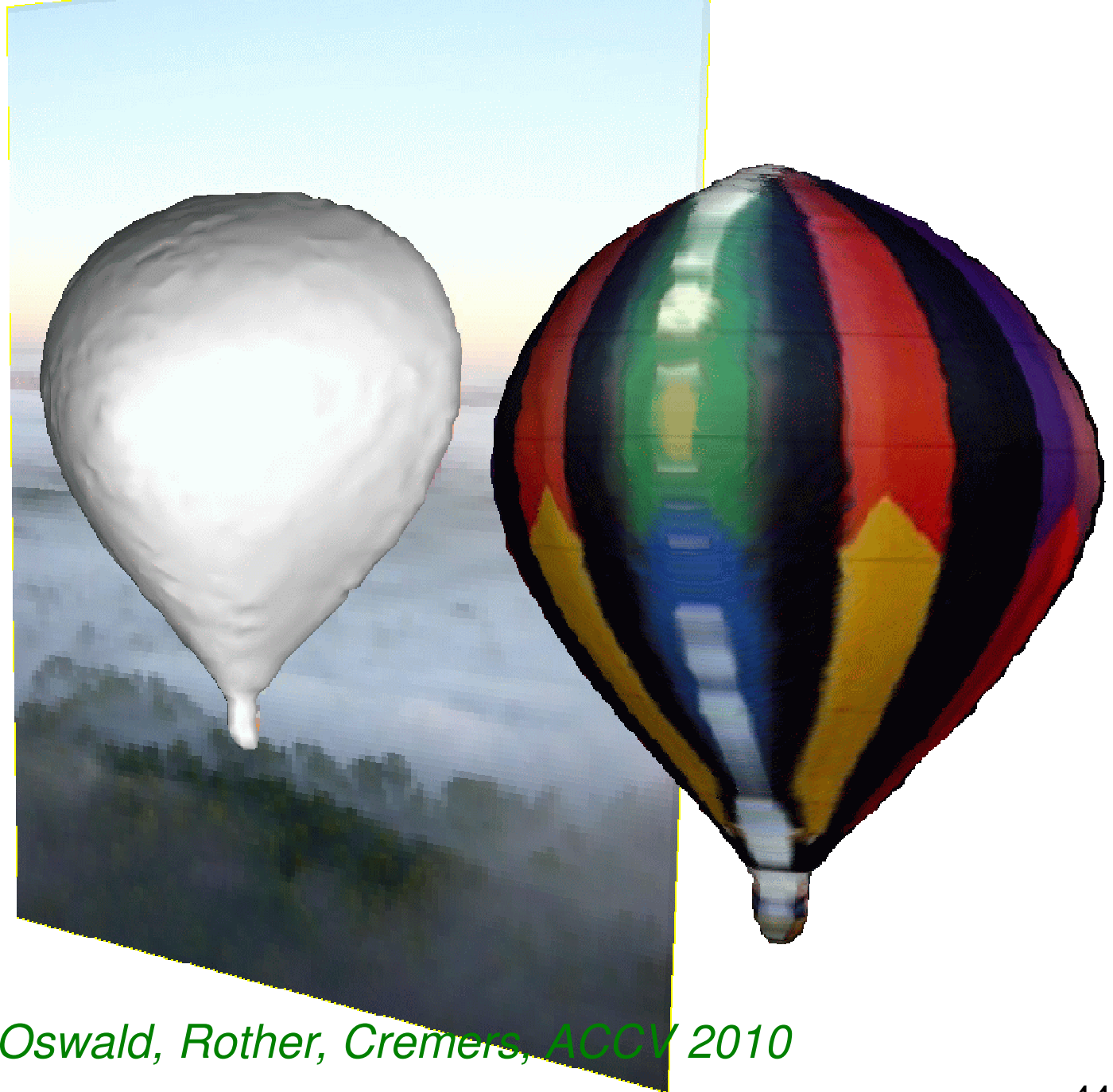
where $u \in BV(\mathbb{R}^3, [0, 1])$ denotes the (relaxed) volume occupancy.

Proposition: The relaxed problem is convex.

Toeppe, Oswald, Rother, Cremers, ACCV 2010

In collaboration with Microsoft Research Cambridge

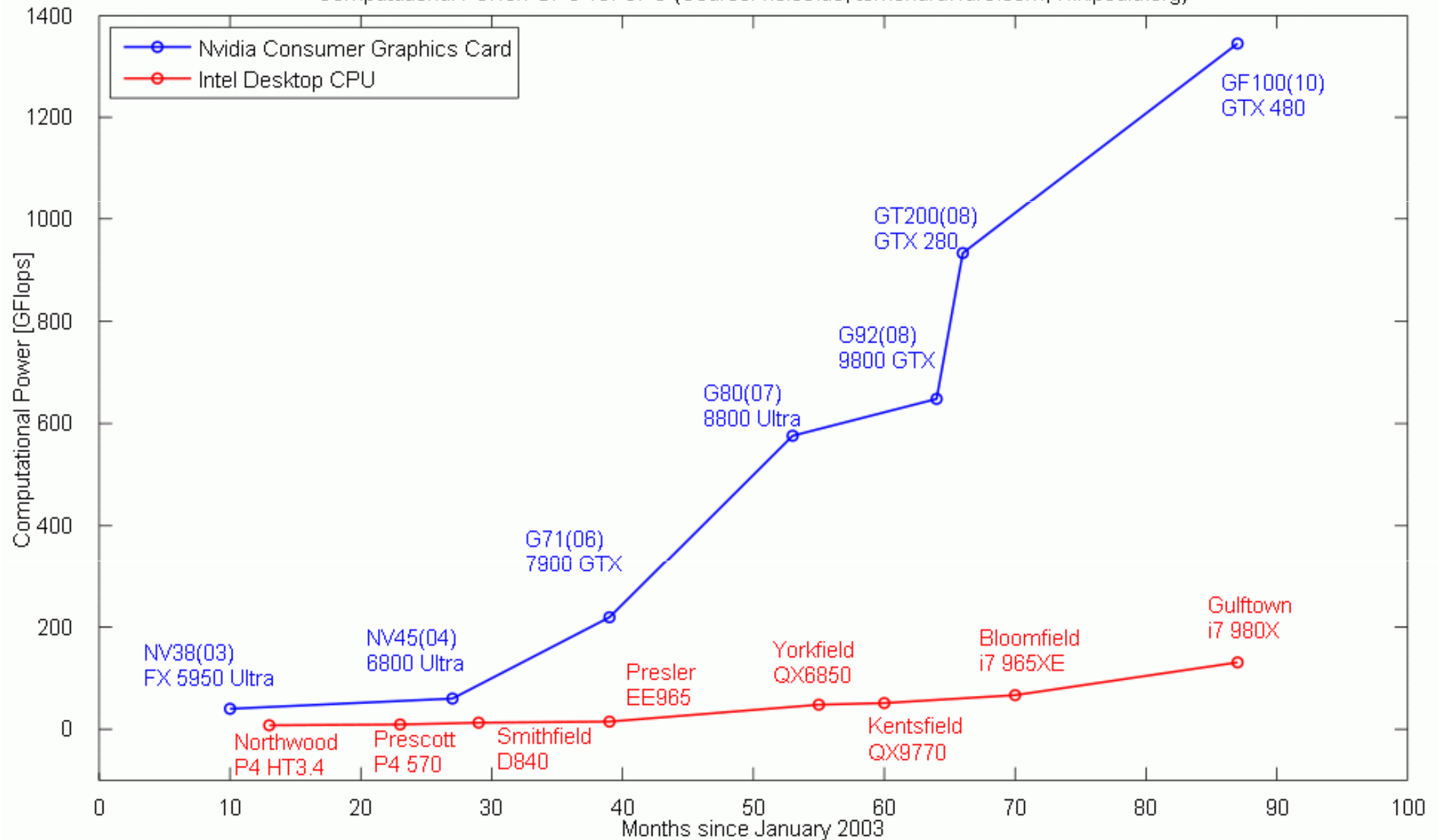
Variational single view reconstruction

CV 2010 Tutorial
Variational Methods in Computer Vision

Toeppe, Oswald, Rother, Cremers, ACCV 2010

Speedups in GPU computation: GFlops comparison

Computational Power: GPU vs. CPU (Source: heise.de, tomshardware.com, wikipedia.org)



Variational single view reconstruction

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Variational Methods in Computer Vision

Input



Reconstruction



+30% volume



+40% volume

Computation time approximately 1 second (on GPU).

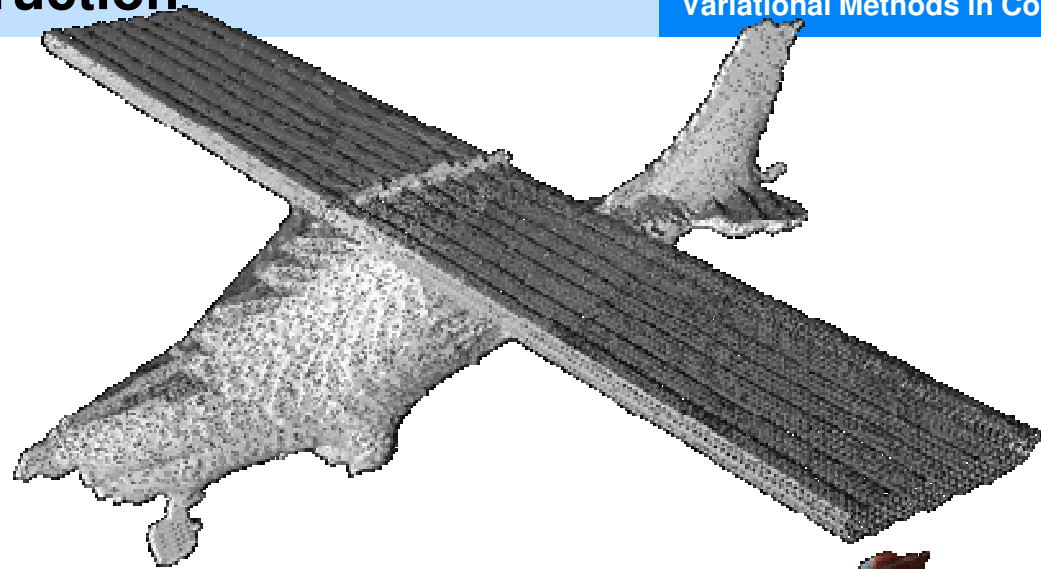
Toeppe, Oswald, Rother, Cremers, ACCV 2010

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Variational single view reconstruction

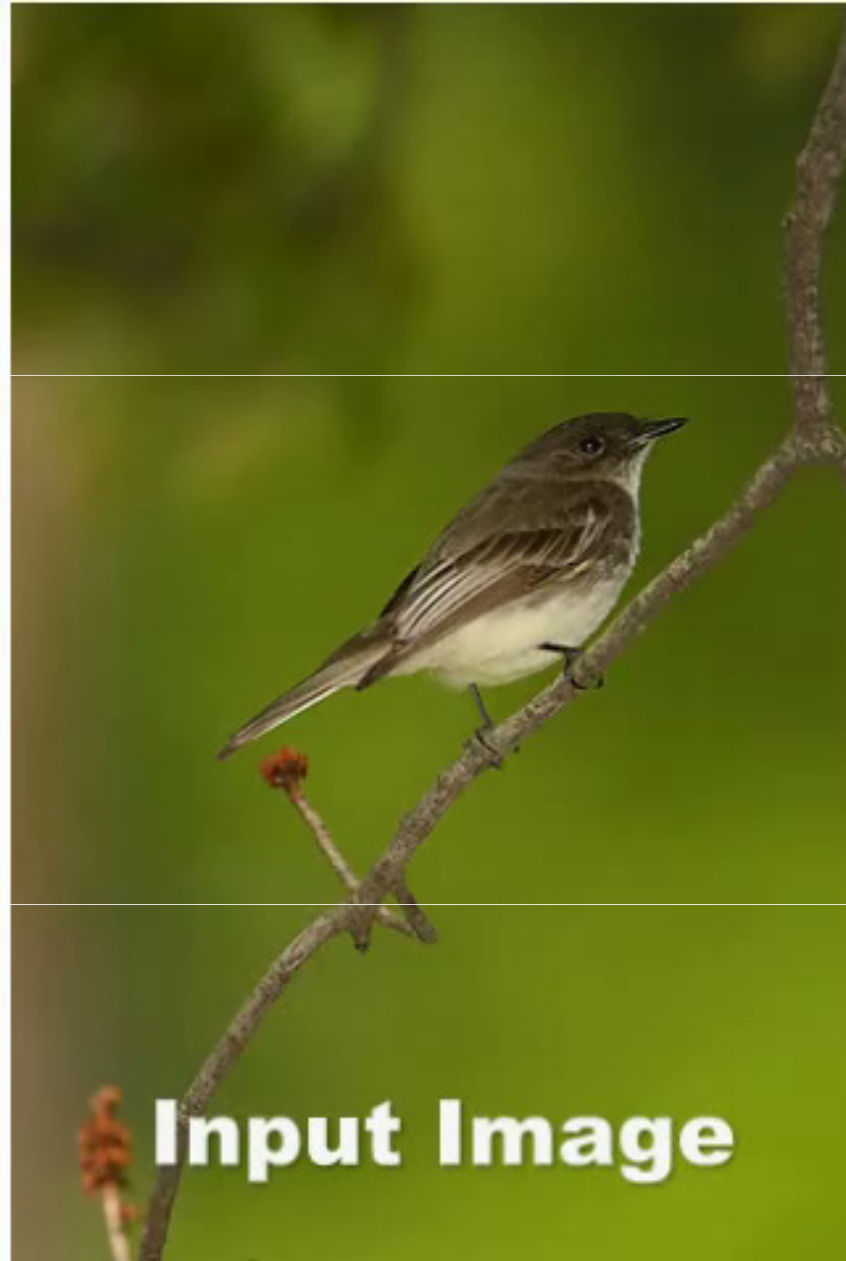
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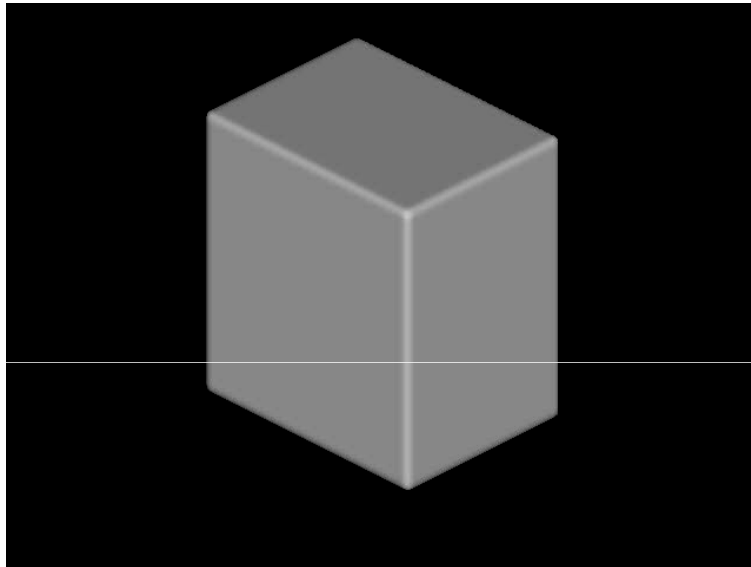
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Variational single view reconstruction

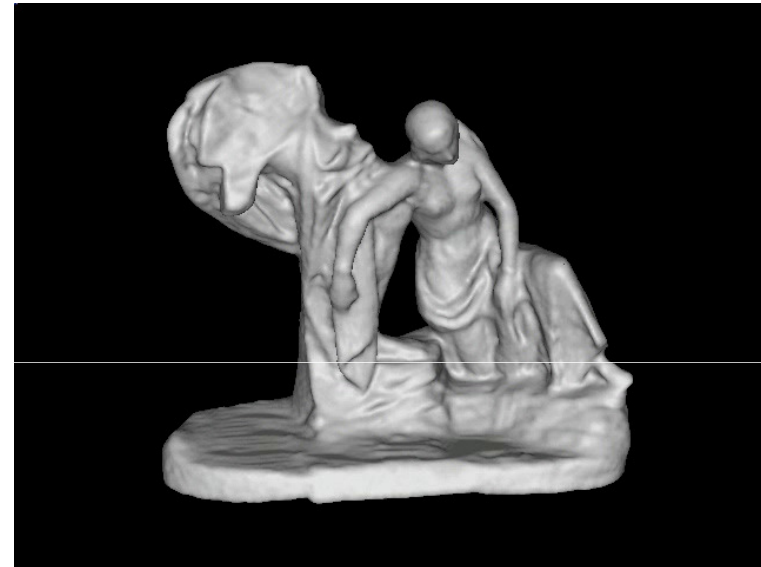
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Multiview reconstruction
via convex relaxation



Stereo & silhouettes via convex
optimization & convex constraints



Superresolution texture



Single view reconstruction