Note: This exercise sheet is made to help you refresh some important concepts of Linear Algebra that are relevant for this course. It is not meant to be a homework assignment. Nevertheless being familiar and having these concepts fresh in mind will help you and save you time when studying the topics of the course.

Exercise 1: Warm up

a) What multiple of \( a = (1,1,1) \) is closest to the point \( b = (2,4,4) \)? Find also the closest point to \( a \) on the line through \( b \).

b) Prove that the trace of \( P = aa^T/a^Ta \) always equals 1.

c) Show that the length of \( Ax \) equals the length of \( A^Tx \) if \( AA^T = A^TA \).

d) Which \( 2 \times 2 \) matrix projects the \( x,y \) plane onto the line \( x + y = 0 \)?

Exercise 2: Determinants

a) If a square matrix \( A \) has determinant \( \frac{1}{2} \), find \( \det(2A) \), \( \det(-A) \), \( \det(A^2) \) and \( \det(A^{-1}) \).

b) Find the determinants of

\[
A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & 4 & 8 & 8 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad U^T \text{ and } U^{-1}
\]

Exercise 3: Eigenvalues and Eigenvectors

a) Find the eigenvalues and eigenvectors of

\[
A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad \text{their traces and their determinants.}
\]

b) Using the characteristic polynomial, find the relationship between the trace, the determinants and the eigenvalues of any square matrix \( A \).
c) Diagonalize the unitary matrix $V$ to reach $V = UΛU^*$. All $|λ| = 1$. $V = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 - i \\ 1 + i & -1 \end{bmatrix}$

d) Suppose $T$ is a $3 \times 3$ upper triangular matrix with entries $t_{ij}$. Compare the entries of $T^*T$ and $TT^*$. Show that if they are equal, then $T$ must be diagonal. (All normal triangular matrices are diagonal)

**Exercise 4: Singular Value Decomposition**

a) Find the singular values and singular vectors of $A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$

b) Explain how $USV^T$ expresses $A$ as a sum of $r$ rank-1 matrices: $A = σ_1 u_1 v_1^T + \ldots + σ_r u_r v_r^T$

c) If $A$ changes to $4A$ what is the change in the SVD?
   
   What is the SVD for $A^T$ and for $A^{-1}$?

d) Find the SVD and the pseudoinverse of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$