Mathematics: Recap of Linear Algebra

Let us start with some definitions

**Definition (Inner product).** Let $X$ be a vector space. A mapping $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ is called inner product, if

1. $\langle x_1 + \lambda x_2, y \rangle = \langle x_1, y \rangle + \lambda \langle x_2, y \rangle \quad \forall x_i, y \in X, \lambda \in \mathbb{C}$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x, y \in X$
3. $\langle x, x \rangle \geq 0 \quad \forall x \in X$
4. $\langle x, x \rangle = 0 \iff x = 0$

Two elements $x, y \in X$ are called perpendicular, if $\langle x, y \rangle = 0$.

**Definition (Linear operator).** Let $X$ and $Y$ be vector spaces. A mapping $T : X \to Y$ is called linear, if

$$T(x_1 + \lambda x_2) = T(x_1) + \lambda T(x_2)$$

A common notation is $Tx := T(x)$

**Definition (Eigenvalues and eigenvectors).** Let $T : X \to X$ be a linear operator from a vector space $X$ into itself (an endomorphism). An eigenvector is an element $0 \neq x \in X$ for which there exists a scalar $\lambda \in \mathbb{C}$, such that

$$Tx = \lambda x$$

The scalar $\lambda$ is called eigenvalue.
Exercise 1 (1 point). 1. Show that every matrix $\Phi \in \mathbb{C}^{n \times n}$ is representing an endomorphism on $\mathbb{C}^n$ via

$$(\Phi x)_i = \sum_{j=1}^{n} \phi_{ij} x_j \quad \forall i = 1 \ldots n$$

If we denote the $j$-th column of $\Phi$ by $\phi_j$ we can write $\Phi x = \sum_{j=1}^{n} \phi_j x_j$

2. Calculate the gradient $\nabla f(x) = (\partial_1 f(x), \ldots, \partial_n f(x))^T$ for $f(x) = x^T Ax \ (A \in \mathbb{C}^{n \times n})$.

Exercise 2 (1 point). When not mentioned otherwise we consider the standard inner product on $\mathbb{C}^n$ given by

$$\langle x, y \rangle = x^T \cdot \bar{y} = \sum_{i=1}^{n} x_i \bar{y}_i$$

1. Show that this is indeed an inner product.

2. Given a matrix $A \in \mathbb{C}^{n \times n}$, find the matrix $B$ such that $\forall x, y \in \mathbb{C}^n : \langle Ax, y \rangle = \langle x, By \rangle$.

3. Show that if $A = B$ then all the eigenvalues are real.

4. Show that if $A = B$ and the two eigenvectors $x^1$ and $x^2$ are not orthogonal, it follows $\lambda_1 = \lambda_2$.

Programming: Working with Matlab and triangle meshes

A triangle mesh $\mathcal{M} = (\mathcal{V}, \mathcal{F})$ is a discrete surface embedded into $\mathbb{R}^3$. It consists of a vertex set $\mathcal{V} = \{v_1, \ldots, v_n\}$, and a set of triangles $\mathcal{F} \subset \mathcal{V} \times \mathcal{V} \times \mathcal{V}$ (also called faces in a more general setting). The coordinates of the vertices embedded in $\mathbb{R}^3$ are denoted by $x(v_1), \ldots, x(v_n) \in \mathbb{R}^3$. Note that the vertices of a triangle $t = (u, v, w) \in \mathcal{F}$ are ordered. We define triangles to be identical if they can be transformed into each other by a cyclic permutation, i.e. $(u, v, w) = (v, w, u) = (w, u, v)$ but $(u, v, w) \neq (w, v, u)$.

This programming exercises will introduce you to working with meshes in Matlab.

Exercise 3 (One point for 1./2. and one point for 3./4./5.). Download and expand the file exercise1.zip from the lecture website. Modify the files adjacency.m, incidence.m, facearea.m, meshvolume.m and gaussiancurvature.m to implement the functions as explained below. You can run the script exercise.m to test and visualize your solutions.
1. Given a triangle mesh $\mathcal{M} = (\mathcal{V}, \mathcal{F})$ with $n$ vertices the adjacency matrix $A \in \mathbb{R}^{n \times n}$ is defined by
   \[
   A_{i,j} = \begin{cases} 
   1 & \exists t \in \mathcal{F} : v_i, v_j \in t \\
   0 & \text{otherwise.} 
   \end{cases} 
   \]
   Implement function adjacency that returns the adjacency matrix in sparse format for a given triangle mesh.

2. A mesh $\mathcal{M} = (\mathcal{V}, \mathcal{F})$ with $n$ vertices induces a set of halfedges $\mathcal{H} = \{(u, v) \in \mathcal{V} \times \mathcal{V} : \exists w \in \mathcal{V} : (u, v, w) \in \mathcal{F}\}$. We assume some ordering on the halfedges, i.e. $\mathcal{H} = \{h_1, \ldots, h_m\}$. The vertex-to-halfedge incidence matrix $I \in \mathbb{R}^{m \times n}$ is defined by
   \[
   I_{i,j} = \begin{cases} 
   -1 & h_i = (v_j, \cdot) \\
   1 & h_i = (\cdot, v_j) \\
   0 & \text{otherwise.} 
   \end{cases} 
   \]
   Implement function incidence that returns the vertex-to-halfedge incidence matrix in sparse format for a given triangle mesh.

3. Implement function facearea that returns the area of each triangle as an $\mathbb{R}^{\lvert \mathcal{F} \rvert}$ vector for a given triangle mesh.

4. Implement function meshvolume that returns the volume of a given triangle mesh. Hint: Think about the volume of a tetrahedron constructed from the vertices of a mesh triangle and the origin.

5. The Gaussian curvature at vertex $v$ is given by $\kappa_v = 2\pi - \sum_{t \in \mathcal{F} : v \in t} \theta_{t,v}$, where $\theta_{t,v}$ is the angle of triangle $t$ at vertex $v$. Implement function gaussiancurvature that returns the Gaussian curvature of each vertex as an $\mathbb{R}^{\lvert \mathcal{V} \rvert}$ vector for a given triangle mesh.