Analysis of Three-Dimensional Shapes
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The First Fundamental Form
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Dr. Emanuele Rodolà
rodola@in.tum.de
Room 02.09.058, Informatik IX
Wrap-up

Last time we have introduced the main notions of **differential geometry** that we will be using in this course.

In particular, we showed how to model a 3D shape as a **regular surface**, that is, just a collection of deformed plane patches (called **surface elements**) glued together so as to form something smooth.
Wrap-up

The general idea of this approach is that we wish to analyze shapes according to a simple recipe:

- Consider each point of the shape as belonging to some surface element.
- Each surface element is the image of a known diffeomorphism, namely a parametrization function (or chart) $x : \mathbb{R}^2 \rightarrow \mathbb{R}^3$
- Talking about the surface then corresponds to talking about $x$. 
Wrap-up

In doing so, there are some properties that we naturally expect to be satisfied:

• The local properties of the surface should not depend on the specific choice of a parametrization $x$.
• Since we want to speak about tangent planes, the parametrization should be differentiable.
• Since we know how to do calculus in $\mathbb{R}^n$, we would like to transfer this knowledge to the study of non-Euclidean domains (the surface).
Wrap-up

\[ q = \mathbf{x}(p) = \mathbf{x}(u(t), v(t)) \big|_{t=0} \]

\[ w = \frac{d}{dt} q \bigg|_{t=0} = \frac{d}{dt} \mathbf{x}(u(t), v(t)) \bigg|_{t=0} = \frac{\partial \mathbf{x}}{\partial u} \frac{du}{dt} + \frac{\partial \mathbf{x}}{\partial v} \frac{dv}{dt} = \mathbf{x}_u u' + \mathbf{x}_v v' = \alpha \mathbf{x}_u + \beta \mathbf{x}_v \]

\[ \mathbf{d}x_p ((\alpha, \beta)) = w \]

\[ \mathbf{d}x_p (e_1) = \mathbf{x}_u \]

\[ \mathbf{d}x_p (e_2) = \mathbf{x}_v \]
Wrap-up
Wrap-up

Let us consider a triangle mesh composed of $m$ triangles. Our triangle-based parametrization is then described by the charts $x_j : \mathbb{R}^2 \to \mathbb{R}^3$ for $j = 1, \ldots, m$

$$x_j(u, v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1})$$

with $u \in [0, 1], v \in [0, 1 - u]$.
Measuring lengths and areas

We are going to introduce a tool for measuring **metric properties** of a surface, such as *lengths*, *areas*, and *integrals* of scalar functions.
First fundamental form

The quadratic form \( I_p : T_p(S) \rightarrow \mathbb{R} \) given by

\[
I_p(w) = \langle w, w \rangle_p = \|w\|^2
\]

is called the **first fundamental form** of the regular surface \( S \) at \( p \).

We write \( \langle \cdot, \cdot \rangle_p \) to remind ourselves that we are computing the inner product on the tangent plane at \( p \).

The first fundamental form is, intuitively, the expression of how the surface \( S \) “inherits” the natural inner product of \( \mathbb{R}^3 \).
First fundamental form

Let us denote by \( \{ \mathbf{x}_u, \mathbf{x}_v \} \) the basis associated to a parametrization \( \mathbf{x}(u, v) \) at \( p \) (thus, \( \{ \mathbf{x}_u, \mathbf{x}_v \} \) spans the tangent plane \( T_p(S) \)).

Any vector \( w \in T_p(S) \) is the tangent vector to a curve \( \alpha(t) = \mathbf{x}(u(t), v(t)) \) which lies on the surface, with \( t \in (-\varepsilon, \varepsilon) \) and \( p = \alpha(0) \).

Then we can write:

\[
I_p(w) = I_p(\alpha'(0)) = \left\langle \alpha'(0), \alpha'(0) \right\rangle_p = \left\langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_u u' + \mathbf{x}_v v' \right\rangle_p
\]

\[
= \left\langle \mathbf{x}_u, \mathbf{x}_u \right\rangle_p (u')^2 + 2 \left\langle \mathbf{x}_u, \mathbf{x}_v \right\rangle_p u' v' + \left\langle \mathbf{x}_v, \mathbf{x}_v \right\rangle_p (v')^2
\]

\[
= E(u')^2 + 2F u' v' + G(v')^2
\]
First fundamental form

\[ I_p(w) = E(u')^2 + 2Fu'v' + G(v')^2 \]

\[ E = \langle x_u, x_u \rangle_p \]
\[ F = \langle x_u, x_v \rangle_p \]
\[ G = \langle x_v, x_v \rangle_p \]

\[ \iff \]

\[ I_p(w) = (u' \ v') \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \]

also called (Riemannian) **metric** tensor

\( E, F, \text{ and } G \) are often called the “coefficients” or “components” of the first fundamental form. These coefficients play important roles in many intrinsic quantities of the surface.

By letting \( p \) run in the neighborhood defined by \( x(u,v) \) we obtain smooth functions \( E(u,v), \ F(u,v), \ G(u,v) \). A manifold together with this smooth inner product is called a **Riemannian manifold**.
First fundamental form

\[ I_p(w) = E(u')^2 + 2Fu'v' + G(v')^2 \]

\[ E = \langle x_u, x_u \rangle_p \]
\[ F = \langle x_u, x_v \rangle_p \]
\[ G = \langle x_v, x_v \rangle_p \]

\[ I_p(w) = (u' \ v') \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \]

this is often called just \( g \)

We have seen that the differential map associated to \( x \) is represented by the Jacobian matrix:

\[ \text{D}x = \begin{pmatrix} \vdots & \vdots \\ x_u & x_v \\ \vdots & \vdots \end{pmatrix} \]

Then, it is easy to see that:

\[ g = \text{D}x^T \text{D}x = \begin{pmatrix} \langle x_u, x_u \rangle & \langle x_u, x_v \rangle \\ \langle x_v, x_u \rangle & \langle x_v, x_v \rangle \end{pmatrix} \]
Parametrizations

\[ I_p(w) = \begin{pmatrix} u' & v' \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \]

- \( E = \langle x_u, x_u \rangle_p \)
- \( F = \langle x_u, x_v \rangle_p \)
- \( G = \langle x_v, x_v \rangle_p \)

- **generic parametrization**
- **orthogonal parametrization** \( F \equiv 0 \)
- **conformal parametrization** \( F \equiv 0, \ E = G \)
- **isometric parametrization** \( F \equiv 0, \ E = G = 1 \)
The confusing example

Consider a plane \( S \subset \mathbb{R}^3 \) passing through \( q_0 \) and containing the orthonormal vectors \( \tilde{w}_1 \) and \( \tilde{w}_2 \).

\[
\tilde{x}(u, v) = q_0 + u\tilde{w}_1 + v\tilde{w}_2 \implies \begin{align*}
\tilde{x}_u &= \tilde{w}_1 \\
\tilde{x}_v &= \tilde{w}_2
\end{align*}
\]

We want to compute the first fundamental form for an arbitrary point \( q \) in \( S \).

Thus, the first fundamental form of \( w \) at \( p \) is \( I_p((\bar{\alpha}, \bar{\beta})) = (\bar{\alpha} \quad \bar{\beta}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} = \bar{\alpha}^2 + \bar{\beta}^2 \)

Euclidean metric tensor

\[
\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
Example 2 (plane)

Consider the previous example, but this time let $\|w_1\| = 1$ and $\|w_2\| = 2$. We are changing the parametrization $x$, but still we expect that the lengths of vectors in $T_p(S)$ do not change (as they are a property of the surface).

Say, for example, that we take the same $(p,w)$ from the previous example.

As before, we have $x_u = w_1$, $x_v = w_2$, and then $g = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

We can now compute $I_p((\alpha, \beta)) = (\alpha \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \beta) = \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \frac{\tilde{\beta}}{2} & \frac{\tilde{\alpha}}{2} \end{pmatrix} = \tilde{\alpha}^2 + \tilde{\beta}^2$

previous example: $w = \tilde{\alpha}x_u + \tilde{\beta}x_v$
this example: $w = \alpha x_u + \beta x_v$

The two bases, and thus the coefficients for $w$ are different in the two examples.

$\alpha x_u + \beta x_v = \tilde{\alpha}x_u + \tilde{\beta}x_v$
Example 3 (plane)

Let’s make it more interesting and let $\|w_1\| = 1$, $\|w_2\| = 1$, and $\langle w_1, w_2 \rangle = \frac{1}{\sqrt{2}}$. Again, we expect that the length of $w$ does not change.

Once again, we have $x_u = w_1$, $x_v = w_2$, and now $g = \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix}$.

Even though the coefficients in $g$ are different, again we expect the first fundamental form to be the same as before.

So we get $I_p((\alpha, \beta)) = (\alpha \quad \beta) \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix} (\alpha \quad \beta) = (\tilde{\alpha} - \tilde{\beta} \quad \sqrt{2}\tilde{\beta}) \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix} (\tilde{\alpha} - \tilde{\beta} \quad \sqrt{2}\tilde{\beta}) = \tilde{\alpha}^2 + \tilde{\beta}^2$.
Example 4 (cylinder)

\[ x(u, v) = (\cos u, \sin u, v) \]

\[ U = \{(u, v) \in \mathbb{R}^2; \ 0 < u < 2\pi, \ -\infty < v < \infty \} \]

\[ x_u = (-\sin u, \cos u, 0), \quad x_v = (0, 0, 1) \]

\[ E = \sin^2 u + \cos^2 u = 1 \]

\[ F = 0 \]

\[ G = 1 \]

\[ g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

We notice that the plane and the cylinder behave locally in the same way, since their first fundamental forms are equal.

In other words, plane and cylinder are locally isometric. We will discuss about local isometries more in detail at a later time.
Example 5a (sphere)

\[ \mathbf{x} : (0, 2\pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}^3 \]

\[ \mathbf{x}(u, v) = \begin{pmatrix} \cos(u) \cos(v) \\ \sin(u) \cos(v) \\ \sin(v) \end{pmatrix} \]

\[ \mathbf{Dx} = \begin{pmatrix} \vdots & \vdots \\ \mathbf{x}_u & \mathbf{x}_v \\ \vdots & \vdots \end{pmatrix} = \begin{pmatrix} -\sin(u) \cos(v) & -\cos(u) \sin(v) \\ \cos(u) \cos(v) & -\sin(u) \sin(v) \\ 0 & \cos(v) \end{pmatrix} \]

\[ g = \mathbf{Dx}^T \mathbf{Dx} = \begin{pmatrix} \cos^2(v) & 0 \\ 0 & 1 \end{pmatrix} \]

From this example it becomes evident that the coefficients \( E, F, G \) are indeed differentiable functions \( E(u,v), F(u,v), G(u,v) \).

Thus, if \( \mathbf{w} = \alpha \mathbf{x}_u + \beta \mathbf{x}_v \) is the tangent vector to the sphere at point \( \mathbf{x}(u,v) \), then its squared length is given by \( |\mathbf{w}|^2 = I(\mathbf{w}) = \alpha^2 \cos^2(v) + \beta^2 \).
Example 5b (sphere)

\[
y : \mathbb{R}^2 \rightarrow \mathbb{R}^3
\]

\[
y(\tilde{u}, \tilde{v}) = \frac{1}{\tilde{u}^2 + \tilde{v}^2 + 1} \begin{pmatrix} 2\tilde{u} \\ 2\tilde{v} \\ \tilde{u}^2 + \tilde{v}^2 - 1 \end{pmatrix}
\]

\[
Dy(\tilde{u}, \tilde{v}) = \frac{2}{(\tilde{u}^2 + \tilde{v}^2 + 1)^2} \begin{pmatrix} \tilde{v}^2 - \tilde{u}^2 + 1 & 2\tilde{u}\tilde{v} \\ 2\tilde{u}\tilde{v} & \tilde{u}^2 - \tilde{v}^2 + 1 \\ 2\tilde{u} & 2\tilde{v} \end{pmatrix}
\]

\[
g = Dy^T Dy
\]

The result is probably going to look not very nice.

In general, from a computational point of view it is much more convenient to plug in the values for \( \tilde{u}, \tilde{v} \) directly in \( Dy(\tilde{u}, \tilde{v}) \), and only then compute \( g \).
Length of a curve

By knowing the first fundamental form, we can treat metric questions on a regular surface without further references to the ambient space.

The arc-length of a curve \( \alpha : (0, T) \to S \) is given by:

\[
s(t) = \int_0^t \left\| \alpha'(x) \right\| dx = \int_0^t \sqrt{I(\alpha'(x))} dx
\]

Where \( I(\alpha'(x)) \) is the first fundamental form.

Thus, if \( \alpha(t) = \mathbf{x}(u(t), v(t)) \) is contained in a surface element parametrized by \( \mathbf{x}(u,v) \), we can compute the length as:

\[
s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} \, dt
\]

Remember that \( E, F, G \) are actually functions of \((u,v)\), so in general they are changing along the curve.
Length of a curve

\[ \alpha(t) = \mathbf{x}(u(t), v(t)) \]

Length \( s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} \, dt \)
Length of a curve

**Pitfall:** Notice that we are talking about length of curves **within** surface elements, thus in general this construction cannot be used to compute distances among *any two* given surface points.

In fact, we wrote:

Thus, if \( \alpha(t) = x(u(t), v(t)) \) is contained in a surface element parametrized by \( x(u,v) \), we can compute the length as:

\[
s(t) = \int_0^t \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} \, dt
\]
Arc length element

Length \( s(t) = \int_0^t \|\alpha'(x)\| \, dx \)

The first fundamental theorem of calculus gives us:

\[
\frac{ds}{dt} = \|\alpha'(t)\|
\]

A more compact notation for the length of a curve can then be defined as:

\[
\text{length } (\alpha) = \int_\alpha ds
\]

In terms of the metric tensor, the arc length element \( ds \) is given by:

\[
ds = \sqrt{Edx^2 + 2Fdu^2 + Gdv^2} \, dt
\]
Area of a region

The first fundamental form can be employed to compute the area of a bounded region \( R \) of a regular surface \( S \). If \( R \subseteq S \) is contained in the image of the parametrization \( \mathbf{x}: U \subseteq \mathbb{R}^2 \to S \), the area of \( R \) is defined by

\[
A(R) = \iint_{Q} \| \mathbf{x}_u \times \mathbf{x}_v \| \, dudv, \quad Q = \mathbf{x}^{-1}(R)
\]
The area of a region on the surface is defined as the sum of the areas of parallelograms tangent to that surface region.

\[
A(R) = \iint_Q \| \mathbf{x}_u \times \mathbf{x}_v \| \, dudv, \quad Q = \mathbf{x}^{-1}(R)
\]
Area of a region

\[ A(R) = \iiint_Q \| \mathbf{x}_u \times \mathbf{x}_v \| dudv, \quad Q = x^{-1}(R) \]

\[ g = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \]

Observe that:

\[ \| \mathbf{x}_u \times \mathbf{x}_v \|^2 = \| \mathbf{x}_u \|^2 \| \mathbf{x}_v \|^2 \sin^2 \omega = \| \mathbf{x}_u \|^2 \| \mathbf{x}_v \|^2 (1 - \cos^2 \omega) = \| \mathbf{x}_u \|^2 \| \mathbf{x}_v \|^2 - \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2 \]

We can rewrite:

\[ \| \mathbf{x}_u \times \mathbf{x}_v \| = \sqrt{\| \mathbf{x}_u \|^2 \| \mathbf{x}_v \|^2 - \langle \mathbf{x}_u, \mathbf{x}_v \rangle^2} = \sqrt{EG - F^2} = \sqrt{\det g} \]

And so we come to the more compact writing:

\[ A(R) = \iiint_Q \sqrt{\det g} dudv \]
Area element

Similarly to the arc length case, we can define the area element $da$ as:

$$da = \sqrt{\det g} \, du \, dv$$

And then we can use the following notation:

$$A(R) = \int_R da$$

The area element is also called (Riemannian) volume form. In the case of 2-dimensional manifolds (our case), volume corresponds to area.
We have obtained two alternative expressions for measuring lengths and areas: one is defined in parameter space, the other is defined directly on the surface. The expressions are as follows:

- Parameter space: 
  \[ A(R) = \int_{R} da = \sqrt{\text{det} g_{uv}} \]
  \[ Q = x^{-1}(R) \]
  \[ \text{length}(\alpha) = \int_{0}^{T} ds = \sqrt{Edu^2 + 2Fdu dv + Gdv^2} dt \]
  \[ \text{length}(\alpha(t)) = \int_{0}^{T} ds = \sqrt{Edu^2 + 2Fdu dv + Gdv^2} dt \]

- Surface: 
  \[\int_{\partial R} da = \int_{Q} \| x_u \times x_v \| dudv \]
  \[\int_{\partial R} ds = \int_{Q} \sqrt{\| x_u \times x_v \|^2} dudv \]

The equations above represent the area and length calculations in both parameter and surface spaces, respectively.
Integral of a function

We can follow a similar approach to compute the integral of a function defined over the surface, $f : S \rightarrow \mathbb{R}$.

Let us use our newly introduced notation:

$$\int_R f(x) dx$$

Analogously to the previous slide, we get to the definition:

$$\int_R f(x) dx = \int \int_Q f(x(u, v)) \sqrt{\text{det} \ g(u, v)} \ du \ dv, \quad Q = x^{-1}(R)$$

Generalizes the substitution rule in classical multivariate calculus.

$$\int_{\phi(U)} f(v) dv = \int_{U} f(\phi(u)) |\text{det}(D \phi)(u)| \ du.$$
Discretization: chart

Let us consider a triangle mesh composed of $m$ triangles. Our triangle-based parametrization is then described by the charts $\mathbf{x}_j : \mathbb{R}^2 \to \mathbb{R}^3$ for $j = 1, \ldots, m$

$$\mathbf{x}_j(u, v) = \mathbf{x}_{j,1} + u(\mathbf{x}_{j,2} - \mathbf{x}_{j,1}) + v(\mathbf{x}_{j,3} - \mathbf{x}_{j,1})$$

with $u \in [0, 1], v \in [0, 1 - u]$.
Discretization: metric tensor

\[ x_j(u, v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1}) \]

We simply have:

\[ x_u = x_{j,2} - x_{j,1} = e_{21} \]
\[ x_v = x_{j,3} - x_{j,1} = e_{31} \]

The coefficients for the metric tensor / first fundamental form are thus given by:

\[
g_j = \begin{pmatrix} E_j & F_j \\ F_j & G_j \end{pmatrix} = \begin{pmatrix} \langle x_u, x_u \rangle & \langle x_u, x_v \rangle \\ \langle x_v, x_u \rangle & \langle x_v, x_v \rangle \end{pmatrix} = \begin{pmatrix} \|e_{21}\|^2 & \langle e_{21}, e_{31} \rangle \\ \langle e_{21}, e_{31} \rangle & \|e_{31}\|^2 \end{pmatrix}
\]
Discretization: area element

\[ x_j(u, v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1}) \]

Let us compute the area of the triangle by applying our definition of area of a region:

\[
\int_{T_j} da = \int_0^1 \int_0^{1-u} \sqrt{\det g_j} \, du \, dv = 2A(T_j) \int_0^1 \int_0^{1-u} du \, dv = 2A(T_j) \frac{1}{2} = A(T_j)
\]
\[ x_j(u, v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1}) \]

\[ \int_{T_j} f \, da = \int_0^1 \int_0^{1-u} f(\mathbf{x}(u, v)) \sqrt{\det g_j} \, dudv \]

\[ = \int_0^1 \int_0^{1-u} f(x_{j,1})(1 - u - v) + f(x_{j,2})u + f(x_{j,3})v \sqrt{\det g_j} \, dudv \]

\[ = \frac{1}{6}(f(x_{j,1}) + f(x_{j,2}) + f(x_{j,3}))2A(T_j) \]

\[ = \frac{1}{3}(f(x_{j,1}) + f(x_{j,2}) + f(x_{j,3}))A(T_j) \]

\( f : S \rightarrow \mathbb{R} \) behaves **linearly** within each triangle and it is uniquely determined by its values at the vertices of the triangle.

Discretization: integral
Discretization: integral

\[ x_j(u, v) = x_{j,1} + u(x_{j,2} - x_{j,1}) + v(x_{j,3} - x_{j,1}) \]

The integral of \( f \) over a region \( R \subseteq S \) is just the sum of the integrals over each triangle \( T_j \).

\[ \int_R f \, da = \sum_{j=1}^{[R]} \int_{T_j} f \, da \]
Suggested reading

- *Differential geometry of curves and surfaces.* Do Carmo – Chapters 2.5, Appendix 2.B
- *Differential Geometry: Curves – Surfaces – Manifolds.* W. Kühnel – Chapter 3A