Analysis of
Three-Dimensional Shapes
(IN2238, TU München, Summer 2015)

Shapes as Metric Spaces
(20.04.2015)

Dr. Emanuele Rodolà
rodola@in.tum.de
Room 02.09.058, Informatik IX
Space of shapes

Is there something like a “space of shapes”? 
Space of shapes

Is there something like a “space of shapes”? Yes!
What’s it good for?

If we can define a **distance measure** based on point-to-point matches, then we can **solve for the matches** minimizing this distance measure.
What’s it good for?

**Deform** shapes by computing **shortest paths** in this space.
Non-rigid similarity

Rigid similarity

Part of the same metric space

Two different metric spaces

Main idea: Find a representation of the two shapes in a common metric space
Metric spaces

A set $M$ is a metric space if for every pair of points $x, y \in M$ there is a metric (or distance) function $d_M : M \times M \to \mathbb{R}_+ \cup \{0, \infty\}$ such that

- **identity of indiscernibles** $d_M (x, y) = 0 \iff x = y$
- **symmetry** $d_M (x, y) = d_M (y, x)$
- **triangle inequality** $d_M (x, y) \leq d_M (y, z) + d_M (z, x)$ for any $x, y, z \in M$

We will specify a metric space as the pair $(M, d_M)$

Satisfying a subset of these properties leads to the definition of “semi”-metric spaces, “pseudo”-metric spaces, etc.
Examples of metric spaces

\[ X = A \subset \mathbb{R}^k \]  
\[ d_X (x, y) = \| x - y \|_2 \]

\[ X = \text{any set} \]  
\[ d_X (x, y) = \begin{cases} 
0 & \text{if } x = y \\
1 & \text{if } x \neq y
\end{cases} \]

\[ X = \mathbb{R} \]  
\[ d_X (x, y) = |x - y| \]
\[ d_X (x, y) = \log |x - y| \]

\[ X = A \times B \]  
\[ d_X ((a_1, b_1), (a_2, b_2)) = \sqrt{d_A (a_1, a_2)^2 + d_B (b_1, b_2)^2} \]
Isometries

Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces.

A map \(f : X \to Y\) is called distance-preserving if

\[
d_X(x, y) = d_Y(f(x), f(y)) \quad \text{for any } x, y \in X
\]

A bijective, distance-preserving map is called an isometry. Two spaces are isometric if there exists an isometry between them.
Isometries

Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces.

A map \(f : X \to Y\) is called distance-preserving if
\[
d_X(x, y) = d_Y(f(x), f(y)) \quad \text{for any } x, y \in X
\]

A bijective, distance-preserving map is called an isometry. Two spaces are isometric if there exists an isometry between them.

**Exercise:** Show that any isometry is a homeomorphism.

**Exercise:** Isn’t bijectivity redundant?

**Answer:** A surjective, distance-preserving map is called an isometry.

**Exercise:** Show that “being isometric” is an equivalence relation.
Metrics

Let $X$ be a metric space and $\lambda > 0$. The metric space $\lambda X$, which we call $X$ dilated, is the same set $X$ equipped with another distance function $d_{\lambda X}$ defined by

$$d_{\lambda X}(x, y) = \lambda d_X(x, y) \quad \text{for all } x, y \in X$$

If $X$ is a metric space and $Y \subset X$, then a metric on $Y$ can be obtained by the restriction $d_Y = d_X \mid_Y$, such that

$$d_Y(x, y) = d_X(x, y) \quad \text{for all } x, y \in Y$$
Metrics

The distance from a point $x$ to a set $S$ in a metric space $X$ is defined by

$$\text{dist}_X(x, S) = \inf_{y \in S} d_X(x, y)$$

The diameter of a set $S$ in a metric space $X$ is defined by

$$\text{diam}(S) = \sup_{x, y \in S} d_X(x, y)$$
Ambient space

If $X$ is a metric space and $Y \subset X$, then $X$ is called ambient space for $Y$.

Restricting $d_X$ to $d_X \mid_Y$ is the simplest, but not the only way to define a metric on a subset. In many cases it is more natural to consider an intrinsic metric, which is generally not equal to the one restricted from the ambient space.

Example

$S^1$ carries the restricted Euclidean metric $\| \cdot \|$.

An alternative is the (shortest) arc length $\gamma$.

Question: is $(S^1, \| \cdot \|)$ isometric to $(S^1, \gamma)$?
Lipschitz maps

A map $f : X \rightarrow Y$ between metric spaces is called Lipschitz if there exists a $C \geq 0$ such that

$$d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2) \quad \text{for all} \quad x_1, x_2 \in X$$

All Lipschitz maps are continuous (exercise!).

Any suitable value of $C$ is referred to as a Lipschitz constant of $f$. The minimal Lipschitz constant is called the dilatation of $f$, denoted by $\text{dil} f$

$$\text{dil} f = \sup_{x_1, x_2 \in X} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)}$$
Bi-Lipschitz maps

A map $f : X \to Y$ between metric spaces is called bi-Lipschitz if there are positive constants $c$ and $C$ such that

$$cd_x(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq Cd_x(x_1, x_2)$$

for all $x_1, x_2 \in X$

A map with Lipschitz constant $C = 1$ is called nonexpanding.
A first notion of “closeness”

Smooth surfaces in $\mathbb{R}^3$ can be (at least locally) parametrized by a domain $D \subset \mathbb{R}^2$, for example as graphs of smooth functions $h : D \to \mathbb{R}$ or as images of embeddings $p : D \to \mathbb{R}^3$

The two parametrizations determine a homeomorphism from one to the other
A first notion of “closeness”

If the surfaces are “close enough” to each other, the homeomorphism should only slightly change certain quantities such as distances, metric tensors, or their derivatives.

We can say that two spaces have a small distance between them if there is a homeomorphism which “almost preserves” certain geometric characteristics, e.g. the distance.
Lipschitz distance

The idea is to consider two metric spaces $X$ and $Y$ close to each other if there is a homeomorphism $f : X \to Y$ such that

$$\frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \approx 1 \quad \text{for all} \quad x_1, x_2 \in X$$

This definition gives a way to measure relative change between metrics.
Lipschitz distance

The idea is to consider two metric spaces $X$ and $Y$ close to each other if there is a homeomorphism $f : X \rightarrow Y$ such that

$$\frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \approx 1 \quad \text{for all } x_1, x_2 \in X$$

Equivalently, we may require

$$\frac{d_X(x_1, x_2)}{d_Y(f(x_1), f(x_2))} \approx 1 \quad \text{for all } x_1, x_2 \in X$$
Lipschitz distance

Recall that the dilatation of a Lipschitz map \( f \) is defined by

\[
\text{dil} \, f = \sup_{x_1, x_2 \in X} \frac{d_y(f(x_1), f(x_2))}{d_X(x_1, x_2)}
\]

Since we are dealing with homeomorphisms, we get for the inverse map \( f^{-1} \)

\[
\text{dil} \, f^{-1} = \sup_{y_1, y_2 \in Y} \frac{d_X(f^{-1}(y_1), f^{-1}(y_2))}{d_Y(y_1, y_2)}
= \sup_{x_1, x_2 \in X} \frac{d_X(x_1, x_2)}{d_Y(f(x_1), f(x_2))}
\]
Lipschitz distance

Let us consider the maximum relative error that we get when mapping via $f$

$$
\varepsilon(X, Y, f) = \max \left\{ \sup_{x_1, x_2 \in X} \frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} , \sup_{x_1, x_2 \in X} \frac{d_X(x_1, x_2)}{d_Y(f(x_1), f(x_2))} \right\}
$$

$$
= \max \{ \text{dil}(f), \text{dil}(f^{-1}) \}
$$

We are interested in maps yielding a relative error $\varepsilon(X, Y, f) \approx 1$. This corresponds to requiring

$$
\inf_{f: X \to Y} \log \varepsilon(X, Y, f) \to 0
$$
Lipschitz distance

The **Lipschitz distance** between two metric spaces $X$ and $Y$ is defined by

$$d_L(X, Y) = \inf_{f: X \to Y} \log(\max\{\text{dil}(f), \text{dil}(f^{-1})\})$$

The infimum is taken over all homeomorphisms such that $f$ and $f^{-1}$ are Lipschitz maps (bi-Lipschitz homeomorphisms).

$d_L$ is a metric on the space of isometry classes of compact metric spaces.
Lipschitz distance

The **Lipschitz distance** between two metric spaces $X$ and $Y$ is defined by

$$d_{\mathcal{L}}(X,Y) = \inf_{f:X \rightarrow Y} \log(\max \{\text{dil}(f), \text{dil}(f^{-1})\})$$

The infimum is taken over all homeomorphisms such that $f$ and $f^{-1}$ are Lipschitz maps (bi-Lipschitz homeomorphisms).

$d_{\mathcal{L}}$ is a metric on the space of isometry classes of compact metric spaces.

We set $d_{\mathcal{L}}(X,Y) = \infty$ if there are no bi-Lipschitz homeomorphisms from $X$ to $Y$.

*Thus, the Lipschitz distance is not suitable for comparing metric spaces that are not bi-Lipschitz homeomorphic.*
Lipschitz distance

\[ d_{\mathcal{L}}(X, Y) = \inf_{f:X \to Y} \log(\max \{\text{dil}(f), \text{dil}(f^{-1})\}) \]

Non-negativity \( (d_{\mathcal{L}}(X, Y) \geq 0) \)

\( f: X \to Y \) is a homeomorphism, therefore either \( \text{dil}(f) \geq 1 \) or \( \text{dil}(f^{-1}) \geq 1 \) \( (f \) and \( f^{-1} \) cannot both decrease the distances).

Symmetry \( (d_{\mathcal{L}}(X, Y) = d_{\mathcal{L}}(Y, X)) \)

Trivial
Lipschitz distance

\[ d_L(X, Y) = \inf_{f: X \to Y} \log(\max \{ \text{dil}(f), \text{dil}(f^{-1}) \}) \]

Triangle inequality \( (d_L(X, Z) \leq d_L(X, Y) + d_L(Y, Z)) \)

\[ f : X \to Y, \ g : Y \to Z \quad \iff \quad g \circ f : X \to Z \]

bi-Lipschitz homeomorphisms \quad bi-Lipschitz homeomorphism

\[ \text{dil}(g \circ f) \leq \text{dil}(f) \cdot \text{dil}(g) \]

**Exercise**: Prove the above facts.

Hence \( \log(\text{dil}(g \circ f)) \leq \log(\text{dil}(f)) + \log(\text{dil}(g)) \), and similarly for \( f^{-1} \circ g^{-1} \)

This implies \( d_L(X, Z) \leq d_L(X, Y) + d_L(Y, Z) \)
Lipschitz distance

\[ d_L(X,Y) = \inf_{f:X \to Y} \log(\max \{\text{dil}(f), \text{dil}(f^{-1})\}) \]

Identity of indiscernibles \( (d_L(X,Y) = 0 \iff X = Y) \)

\[ X = Y \implies d_L(X,Y) = 0 \]

\( X \) and \( Y \) are isometric by assumption, thus substituting an isometry \( f : X \to Y \) in the definition yields \( d_L(X,Y) = 0 \)

\[ d_L(X,Y) = 0 \implies X = Y \]

More complicated (requires compactness)
Lipschitz distance

**Disadvantage:** It requires spaces to be homeomorphic.

\[ d_L(X,Y) = \infty \]
Lipschitz distance

**Disadvantage**: Even for two homeomorphic spaces, it can happen that our intuitive notion of “similarity” is not realized by a homeomorphism.

Any homeomorphism (in fact, any continuous map) from $X$ to $Y$ essentially distorts distances between some points.

Intuitively, we see that the distance between $X$ and $Y$ should be small because each of them is contained in a small neighborhood of the other in $\mathbb{R}^3$. 
The **Hausdorff distance** between two compact subsets \( X, Y \subset (Z, d_Z) \) is defined by

\[
d^Z_H(X, Y) = \max \left\{ \sup_{x \in X} \text{dist}_Z(x, Y), \sup_{y \in Y} \text{dist}_Z(y, X) \right\}
\]

\( d^Z_H \) is a semi-metric on the space of compact subsets of a metric space.
**Hausdorff distance**

The **Hausdorff distance** between two compact subsets $X, Y \subseteq (Z, d_Z)$ is defined by

$$d_H^Z (X, Y) = \max \left\{ \sup_{x \in X} \text{dist}_Z (x, Y), \sup_{y \in Y} \text{dist}_Z (y, X) \right\}$$

$d_H^Z$ is a semi-metric on the space of compact subsets of a metric space.
Hausdorff distance

The **Hausdorff distance** between two compact subsets \( X, Y \subset (Z, d_Z) \) is defined by

\[
d_{\mathcal{H}}^Z(X, Y) = \max \left\{ \sup_{x \in X} \text{dist}_Z(x, Y), \sup_{y \in Y} \text{dist}_Z(y, X) \right\}
\]

\( d_{\mathcal{H}}^Z \) is a semi-metric on the space of compact subsets of a metric space.

Being a “semi”-metric means that \( d_{\mathcal{H}}^Z (X, Y) = 0 \iff X = Y \) does not hold.

Note that a difference in a single point can make \( d_{\mathcal{H}}^Z \) arbitrarily large!
Gromov-Hausdorff distance

Can we define a Hausdorff distance between metric spaces?

The general idea is to embed the two metric spaces $X$ and $Y$ into a new metric space $Z$, and compute the Hausdorff distance on the resulting embeddings.

We proceed by requiring $d_{\mathcal{GH}}(X,Y) < r$ for $r > 0$ if and only if there exists a metric space $Z$ and subspaces $X', Y' \subseteq Z$ which are isometric to $X$ and $Y$, and such that $d_{\mathcal{H}}(X',Y') < r$. 
We will indeed define $d_{\mathcal{GH}}(X,Y)$ as the minimum $r$ for which such $Z$, $X'$ and $Y'$ exist.
The **Gromov-Hausdorff distance** between two metric spaces $X$ and $Y$ is defined by

$$d_{GH} (X,Y) = \inf_{Z,f,g} d^Z_H (f(X), g(Y))$$

The infimum is taken over all ambient spaces $Z$ and isometric embeddings (distance preserving) $f : X \to Z$, $g : Y \to Z$.

$d_{GH}$ is a metric on the space of isometry classes of compact metric spaces.

All ambient metric spaces $Z$ is indeed a huge class of metric spaces!
Example: rigid isometries

Let us consider the case in which \((X, d_X)\) and \((Y, d_Y)\) are subsets of a larger metric space (this brings us back to the Hausdorff case). For example, take \(X, Y \subset (\mathbb{R}^3, \|\cdot\|)\).

\[
d_{GH}(X, Y) = \inf_{Z, f, g} d_H^Z(f(X), g(Y))
\]

\[
\downarrow
\]

\[
d_{\text{rigid}}(X, Y) = \inf_{\phi: Z \to Z} d_H^Z(X, \phi(X))
\]

Where \(\phi\) sweeps all rigid isometries of the form \(\phi(\cdot) = R(\cdot) + T\) with \(\det R = \pm 1\)
Example: rigid isometries

Let us consider the case in which \((X, d_X)\) and \((Y, d_Y)\) are subsets of a larger metric space (this brings us back to the Hausdorff case). For example, take \(X, Y \subset (\mathbb{R}^3, \|\cdot\|)\).

\[
d^{\text{rigid}}_{GH} (X, Y) = \inf_{\phi : X \to Y} \sup_{Z} d^Z_H (X, \phi(X))
\]
Coverings

Let \( x \in (X, d_X) \). An open ball of radius \( r > 0 \) centered at \( x \) is defined by

\[
B_X(x, r) = \{ z \in X : d_X(x, z) < r \}
\]

For a subset \( A \) of \( X \), we define

\[
B_X(A, r) = \bigcup_{a \in A} B_X(a, r)
\]

A set \( C \subset X \) is an \( r \)-covering of \( X \) if \( B_X(C, r) = X \)
Coverings

Let \( \{x_1, \ldots, x_n\} \subseteq X \) be a \( r \)-covering of the compact metric space \( (X, d_X) \). Then

\[
d_{GH} (X, \{x_1, \ldots, x_n\}) \leq r
\]

This tells us that “shape samplings” are close to the underlying shapes in the Gromov-Hausdorff sense.
Consistence to sampling

Let $\{x_i\}_{i=1}^{m}$ be a $r$-covering of $X$, and $\{y_j\}_{j=1}^{m'}$ be a $r'$-covering of $Y$. Then

$$\left| d_{\mathcal{G}H}(X, Y) - d_{\mathcal{G}H}(\{x_i\}_{i=1}^{m}, \{y_j\}_{j=1}^{m'}) \right| \leq r + r'$$

$d_{\mathcal{G}H}$ is consistent to sampling

If we have a way to compute $d_{\mathcal{G}H}$ for dense enough (small $r'$) samplings of $X$ and $Y$, then it would give us a good approximation to what happens in the continuous spaces.
Consistence to sampling gives a formal justification for the surface recognition problem from point samples, showing that it is well posed.
Better bounds than \[ \left| d_{\mathcal{GH}}(X,Y) - d_{\mathcal{GH}}(\{x_i\}_{i=1}^m, \{y_j\}_{j=1}^{m'}) \right| \leq r + r' \]

Tighter bounds can be computed depending on the algorithm. Recall for the rigid case

\[ d^\text{rigid}_{\mathcal{GH}}(X,Y) = \inf_{\phi: \mathcal{Z} \to \mathcal{Z}} d^Z_{\mathcal{H}}(X, \phi(X)) \]

For example, there is an algorithm with provable bounds:

\[ d^\text{rigid}_{\mathcal{H}}(X,Y) - (r + r') \leq d^\text{rigid}_{\mathcal{H}}(\{x_i\}_{i=1}^m, \phi(\{y_j\}_{j=1}^{m'})) \leq 10 \left( d^\text{rigid}_{\mathcal{H}}(X,Y) + (r + r') \right) \]

unknown observed (computable) unknown
Optimal coverings

Can we devise an optimal sampling scheme in a metric sense?
Farthest point sampling

Fix \( n \) the number of points we want to have in our final covering \( \mathbf{X}_n \)

We proceed recursively. Given \( \mathbf{X}_{k-1} \), select \( p \in (X, d_X) \) such that

\[
p = \arg \max_{x \in X} d_X(x, \mathbf{X}_{k-1})
\]

In general the maximum is not unique, one could consider all of them or randomly pick one.

Set \( \mathbf{X}_k = \mathbf{X}_{k-1} \cup \{p\} \), and repeat.
Farthest point sampling

Clearly we expect different samplings depending on the starting point $X_1 = \{ q \}$

One way to get a more stable sampling is by setting $X_2 = \{ p, q \}$ such that

$$(p, q) = \arg \max_{(p, q) \in X \times X} d_X (p, q)$$

In other words, select two points attaining $\text{diam}(X)$

Note however, that such a $X_2$ is still in general not unique.
Voronoi decomposition

The sampling $\{x_i\}_{i=1}^n$ represents a region $V_i \subset X$ as a single point $x_i \in X$:

$$V_i(X) = \{x \in X : d_X(x, x_i) < d_X(x, x_j), x_j \neq i \in X\}$$

This region is also known as **Voronoi region**.

The Voronoi decomposition replaces $x \in X$ with the closest point $\tilde{x}(x) \in X$

Its **representation error** can be quantified by

$$\varepsilon(X) = \text{var}\{d_X(x, \tilde{x}(x))\}$$
Optimal sampling

The optimal sampling is $\arg \min_x \varepsilon(X)$

Alternatively, the optimal sampling is the one minimizing the maximum cluster radius

$$\varepsilon_\infty(X) = \max_{i=1, \ldots, n} \max_{x \in V_i} \max_{x_i} d_x(x, x_i)$$

Both error criteria are NP-hard to compute!
**Theorem:** FPS is “almost” optimal, in the sense

\[ \varepsilon_\infty (X_{fps}) \leq 2 \min_X \varepsilon_\infty (X) \]
Farthest point sampling

• The final sampling has progressively increasing density.

• *It is efficient* (provided the chosen metric is efficient to compute). Time complexity is $O(mn)$, where $m = |X|$. It can be reduced using efficient data structures.

• It is worse than optimal sampling by at most a factor of 2.
Suggested reading

The references below contain more than needed for the course, but cover all the key notions we have seen in this class.

- *Numerical geometry of non-rigid shapes* (Bronstein, Bronstein, Kimmel) – Chapters 10.1, 10.2, 10.3
- *A course in metric geometry* (Burago, Burago, Ivanov) – Chapters 1.1 to 1.5, 7.1, 7.2, 7.3