Time-Discrete Geodesics in the Space of Shells
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The Problem

- Finding a 'natural' path between two given shapes
Shapes

- Thin, smooth materials embedded in $\mathbb{R}^3$
- $\delta$ thickness of the material
- 1-to-1-Correspondence between both shapes is given
Elastic and Viscous Materials
Strain

- Measures the relative change of coordinates
- Compression, tension, shear, but not rigid motion
- Strain tensor for infinitesimal deformations:

\[ A = (d\phi(x), d\phi(y), d\phi(z)) \in \mathbb{R}^{3 \times 3} \]
Strain Tensor of $\phi_t$

Eulerian flow field:

$$\nu_t = \dot{\phi}_t \circ \phi_t^{-1}$$

Symmetrized Jacobian:

$$\epsilon[\nu_t] = \frac{1}{2} \left( (D\nu_t)^\top + D\nu_t \right)$$
Strain Tensor of $\phi_t$

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Elastic Energy Density

- $A \in \mathbb{R}^3$ is a strain tensor
- $W(A)$ is the elastic energy density measuring the elastic energy needed to overcome this strain
- $W(R^\top AR) = W(A)$ for all $R \in SO(3)$ invariance under rigid motion
- $W(Id) = 0, W,A(Id) = 0$ The identity is a minimizer
Dissipation

- Dissipation is the density of viscous friction
- Viscous material transforms energy into heat instead of storing it as stress

**Definition**

\[
\text{Diss}(\epsilon[v_t]) = \frac{1}{2} W_{AA}(Id) [\epsilon[v_t], \epsilon[v_t]]
\]
Energy Dissipation

Definition

Energy Dissipation of path $\phi$:

$$
\mathcal{E}[(\phi_t)_{0 \leq t \leq 1}] = \int_0^1 \int_{\phi_t(M^\delta)} \text{Diss}(\varepsilon[v_t]) \, dx \, dt
$$
**Definition**

We define the *shortest* (or most natural) path between two shells as the *path of least energy dissipation*.

\[
\arg \min_{\phi} \mathcal{E}[(\phi_t)_{0 \leq t \leq 1}]
\]
Time Discretization

- Divide interval $t = [0, 1]$ into $K$ sections
- $\tau = \frac{1}{K}$
- $\phi_k = \phi_{k\tau} \circ \phi_{(k-1)\tau}$
- $M_k^\delta = \phi_k(M_{k-1}^\delta)$
Time Discretization

- Divide interval $t = [0, 1]$ into $K$ sections
- $\tau = \frac{1}{K}$
- $\phi_k = \phi_{k\tau} \circ \phi_{(k-1)\tau}^{-1}$
- $\mathcal{M}_k^\delta = \phi_k(\mathcal{M}_{k-1}^\delta)$

Discrete Path Energy

$$\mathcal{E} [\mathcal{M}_0, \ldots, \mathcal{M}_K] = \frac{1}{\tau} \sum_{k=1}^{K} \mathcal{W} [\mathcal{M}_{k-1}^\delta, \phi_k (\mathcal{M}_{k-1}^\delta)]$$
Time Discretization

- Divide interval \( t = [0, 1] \) into \( K \) sections
- \( \tau = \frac{1}{K} \)
- \( \phi_k = \phi_{k\tau} \circ \phi_{(k-1)\tau}^{-1} \)
- \( \mathcal{M}_k^\delta = \phi_k(\mathcal{M}_{k-1}^\delta) \)

Discrete Path Energy

\[
\mathcal{E}[\mathcal{M}_0, \ldots, \mathcal{M}_K] = \frac{1}{\tau} \sum_{k=1}^{K} \mathcal{W}[\mathcal{M}_{k-1}^\delta, \phi_k(\mathcal{M}_{k-1}^\delta)] \\
\mathcal{W}[\mathcal{M}^\delta, \phi(\mathcal{M}^\delta)] = \int_{\mathcal{M}^\delta} W(A[\phi]) \, dx
\]
Time-Discrete Geodesics

Definition

Given two shells $\mathcal{M}_A$ and $\mathcal{M}_B$ we refer to the minimizer $(\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_K)$ of the discrete path energy, with $\mathcal{M}_0 = \mathcal{M}_A$ and $\mathcal{M}_K = \mathcal{M}_B$, as a *discrete geodesic*.

$$\arg \min_{\mathcal{M}_0, \ldots, \mathcal{M}_K} \mathcal{E}[\mathcal{M}_0, \ldots, \mathcal{M}_K]$$

- for large $K$ the discrete geodesic converges to the continuous optimum
Types of Friction

2 types of friction can cause deformations:

- Tangential shear
- Transversal shear
Tangential Shear

- Membrane energy
- Extension and Contraction
- Change of length and areas on the shape
- Measured by change of first fundamental form
First Fundamental Form

- Inner product on the tangent spaces of the shape

\[ \langle v, w \rangle \equiv \bar{I}(v, w) \]

- Pullback of the FFF of \( M \) to \( \bar{M} \)

\[ \bar{I}(A[\phi]v, w) = I(d\phi(v), d\phi(w)) \]
First Fundamental Form

- Inner product on the tangent spaces of the shape

\[ \langle v, w \rangle \triangleq \tilde{I}(v, w) \]

- Pullback of the FFF of $M$ to $\tilde{M}$

\[ \tilde{I}(A[\phi]v, w) = I(d\phi(v), d\phi(w)) \]

\[ v^\top A[\phi]w = v^\top \left( (d\phi)^\top d\phi \right) w \]
Membrane Energy

\[ W_{\text{mem}}(A) [\mathcal{M}, \phi(\mathcal{M})] = \int_{\mathcal{M}} W_{\text{mem}}(A[\phi]) dx \]

\( \mu, \lambda \) are Lamé constants

\( \text{tr}(A) \) measures length distortion,

\( \det(A) \) area distortion

\( \text{linear in } \delta \)
Membrane Energy

\[
\mathcal{W}_{\text{mem}}(A) \left[ \tilde{M}, \phi(\tilde{M}) \right] = \int_{\tilde{M}} W_{\text{mem}}(A[\phi]) \, dx
\]

\[
W_{\text{mem}}(A) = \frac{\mu}{2} \text{tr}(A) + \frac{\lambda}{4} \text{det}(A) - \frac{2\mu + \lambda}{4} \log(\text{det}(A)) - \mu - \frac{\lambda}{4}
\]
Membrane Energy

\[ W_{mem}(A) [\bar{M}, \phi(\bar{M})] = \int_{\bar{M}} W_{mem}(A[\phi])dx \]

\[ W_{mem}(A) = \frac{\mu}{2} tr(A) + \frac{\lambda}{4} \det(A) - \frac{2\mu + \lambda}{4} \log(\det(A)) - \mu - \frac{\lambda}{4} \]

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- \( \mu, \lambda \) are Lamé constants
- \( tr(A) \) measures length distortion, \( det(A) \) area distortion
- linear in \( \delta \)
Ratio of $\mu/\lambda$

- $\lambda$: resistance against area changes
- $\mu$: resistance against length changes
- Top row: $\mu/\lambda > 1$
- Bottom row: $\mu/\lambda < 1$
Transversal Shear

- Bending energy
- Corresponds to isometric deformations
- Change of curvature of the shape
- Measured in change of the shape operator
Shape Operator and Second Fundamental Form

- SFF measures the change in curvature
- Shape operator gives change of normal in certain direction

\[ \bar{I}(\bar{B}v, w) = \bar{\II}(v, w) \]

Pullback of the Shape Operator from \( \mathcal{M} \) to \( \mathcal{\bar{M}} \)

\[ \bar{I}(Bv, w) = \II(d\phi(v), d\phi(w)) \]
Bending Energy

Relative Weingarten map: \( Q[\phi] = \bar{B} - B \)

\[
W_{\text{bend}} [\bar{M}, \phi(\bar{M})] = \int_{\bar{M}} W_{\text{bend}} (Q[\phi]) \, dx
\]
Relative Weingarten map: $Q[\phi] = \bar{B} - B$

**Bending Energy**

$$W_{\text{bend}} \left[ \bar{M}, \phi(\bar{M}) \right] = \int_{\bar{M}} W_{\text{bend}} (Q[\phi]) \, dx$$

$$W_{\text{bend}}(Q) = \|Q\|_F^2$$

where $\|\cdot\|_F$ denotes Frobenius norm
Bending Energy

Relative Weingarten map: \(Q[\phi] = \bar{B} - B\)

\[
\mathcal{W}_{bend} [\bar{\mathcal{M}}, \phi(\bar{\mathcal{M}})] = \int_{\bar{\mathcal{M}}} W_{bend} (Q[\phi]) \, dx
\]

\[
W_{bend}(Q) = \|Q\|_F^2
\]

where \(\| \cdot \|_F\) denotes Frobenius norm

- cubic in \(\delta\)
Deformation Energy

\[ \mathcal{W} [\mathcal{M}, \phi(\mathcal{M})] = (\mathcal{W}_{mem} + \eta\mathcal{W}_{bend}) [\mathcal{M}, \phi(\mathcal{M})] \]

- choose \( \eta = \delta^2 \)

\[ \mathcal{E} [\mathcal{M}_0, \ldots, \mathcal{M}_k] = \frac{1}{\tau} \sum_{k=1}^{K} \mathcal{W} [\mathcal{M}_{k-1}, \phi_k(\mathcal{M}_{k-1})] \]
Deformation Energy

Top: Membrane energy [0, 0.01], Bottom: Bending energy [0, 10]
Physical Tuning

Three tuning parameters:

- $\eta$: material thickness
- $\lambda$: resistance against area changes
- $\mu$: resistance against length changes

All of them have physical meanings
Shell Discretization

- Triangle mesh with fixed connectivity
Constant Strain Triangle Formulation

- symmetric, quadratic form $Q$ on the tangent plane
Constant Strain Triangle Formulation

- symmetric, quadratic form $Q$ on the tangent plane
- evaluation by using three non-parallel vectors (i.e. triangle edges for triangle meshes)
Constant Strain Triangle Formulation

- symmetric, quadratic form $Q$ on the tangent plane
- evaluation by using three non-parallel vectors (i.e. triangle edges for triangle meshes)

$$Q = \frac{1}{8|T|^2} \sum_{i=1}^{3} (Q_j + Q_k - Q_i) t_i \otimes t_i$$
Membrane Energy

Continuous:

\[ \int_{\tilde{\mathcal{M}}} W_{\text{mem}}(A[\phi]) \, dx \]

Discrete:

\[ \sum_{\tilde{T}} W_{\text{mem}}(A_{\tilde{T}}) |\tilde{T}| \]
Membrane Energy

Continuous:
\[
\int_{\mathcal{M}} W_{\text{mem}}(A[\phi]) \, dx
\]

Discrete:
\[
\sum_{\mathcal{T}} W_{\text{mem}}(A_{\mathcal{T}}) \, |\mathcal{T}|
\]

\[
A_{\mathcal{T}} = \frac{1}{8|\mathcal{T}|^2} \sum_{i=1}^{3} \left( \frac{||e_j||^2}{Q_j} + \frac{||e_k||^2}{Q_k} - \frac{||e_i||^2}{Q_i} \right) \quad \bar{t}_i \otimes \bar{t}_i
\]
Bending Energy

Continuous:

\[
\int_{\mathcal{M}} W_{\text{bend}}(\bar{B} - B) \, dx
\]

Discrete:

\[
\sum_{\bar{T}} W_{\text{bend}} (\bar{B}_{\bar{T}} - B_T) |\bar{T}|
\]
**Bending Energy**

Continuous:

\[
\int_{\mathcal{M}} W_{\text{bend}}(\Bar{B} - B) dx
\]

Discrete:

\[
\sum_{\Bar{T}} W_{\text{bend}} (\Bar{B}\Bar{T} - B_T) |\Bar{T}|
\]

\[
Q_i = 2\langle N_j - N_k, e_i \rangle
\]

\[
B_{\Bar{T}} = \frac{1}{8|\Bar{T}|^2} \sum_{i=1}^{3} (\Bar{Q}_j + \Bar{Q}_k - \Bar{Q}_i) \Bar{t}_i \otimes \Bar{t}_i
\]
Nonlinear Optimization Problem

- nonlinear optimization problem in $\mathbb{R}^{3M(K-1)}$
- Initialization by linear interpolation
- Experiments with Quasi-Newton method
- Robust to exchange of input shells
- Mesh independent
Multilevel Optimization
Multilevel Optimization

- 2000 nodes
- 4 resolutions with node reduction by factor 0.3
- 2.5 h to 628 s (≈ 10 min)
Problems and further improvements

- lack of termination for values of $\eta > 1$
- mesh decimation does not account for local deformation strength
- Better results with true Newton method
- Independence of complete correspondence
Any Questions?

Thank you for your attention!