Riemannian shape analysis of 3D Objects
Comparing parameterized surfaces

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25. Juni 2014

based on "A Novel Riemannian Framework for Shape Analysis of 3D Objects" by Sebastian Kurtek et al. (CVPR 2010)
Overview

1. Motivation
2. Defining the distance function
3. The Riemannian distance function
4. Optimization over re-parametrization
5. Experimental results
Goal:

- Define a distance function $d(f_1, f_2)$ which is invariant to
  - global scaling
  - rigid motion
  - re-parametrization

- Compute minimal distance between two shapes w.r.t. re-parametrization and rotation
Definition:
The set of all parameterized surfaces on $D$ is defined by

$$\mathcal{F} = \{ f : D \to \mathbb{R}^3 | \int_D \| f(s) \|^2 ds < \infty \}$$

where $\| \cdot \|$ is the Euclidean norm.
Definition:
The set of all parameterized surfaces on $D$ is defined by

$$\mathcal{F} = \{ f : D \to \mathbb{R}^3 \mid \int_D \| f(s) \|^2 ds < \infty \}$$

where $\| \cdot \|$ is the Euclidean norm.

We use

- $D = [0, 1]^2$ for quadrilateral surfaces
- $D = \mathbb{S}^1$ for hemispherical surfaces
- $D = \mathbb{S}^2$ for closed surfaces.
Definition:
The $L^2$ distance between two $L^2$-measurable points $f_1, f_2 \in \mathcal{F}$ on a domain $D$ is given by

$$d(f_1, f_2) = \left( \int_D \| f_1(s) - f_2(s) \|^2 ds \right)^{1/2}$$

where $\| \cdot \|$ is the Euclidean norm.
**Problem**: This metric is not invariant to re-parametrization of surfaces, as

\[
d(f_1 \circ \gamma, f_2 \circ \gamma) = \left( \int_D \| f_1(\gamma(s)) - f_2(\gamma(s)) \|^2 ds \right)^{\frac{1}{2}}
\]
Problem: This metric is not invariant to re-parametrization of surfaces, as

\[
d(f_1 \circ \gamma, f_2 \circ \gamma) = \left( \int_{D} \|f_1(\gamma(s)) - f_2(\gamma(s))\|^2 ds \right)^{\frac{1}{2}} = \left( \int_{D} \|f_1(\tilde{s}) - f_2(\tilde{s})\|^2 (\text{det} J_{\gamma}(\tilde{s}))^{-1} d\tilde{s} \right)^{\frac{1}{2}} \neq d(f_1, f_2)
\]
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\]

\[
\neq d(f_1, f_2)
\]

Solution:
Introducing so-called q-maps that represent the original functions, so that \(d(q_1, q_2)\) fulfils the desired invariances.
Introducing q-maps

Definition:

Given a differentiable surface $f$, define $\kappa : D \rightarrow \mathbb{R}_+$ as

$$\kappa(s) = \| df_u(s) \times df_v(s) \|$$

where $\{u, v\}$ is an orthonormal basis of $T_s(D)$.

The corresponding **q-map** is then given by $q : D \rightarrow \mathbb{R}^3$,

$$q(s) = \sqrt{\kappa(s)} f(s).$$

As we will see later, the defined metric on q-maps is invariant to re-parametrization.
**Remark:**
\( \kappa(s) \) can be seen as an *infinitesimal volume multiplication factor*, because

**Definition: Surface area**
Let \( f : D \rightarrow \mathbb{R}^3 \) be a parameterized surface. Then the surface area of \( f \) is defined as

\[
A(f) = \int_D \|f_x(s) \times f_y(s)\| \, ds = \int_D \kappa(s) \, ds
\]
Desired invariances in scaling and translation
We can normalize the mappings manually by computing:

\[
f_{scaled}(s) = \frac{f(s)}{\sqrt{\int_D \kappa(s) ds}}
\]
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**Scaling**

\[ f_{scaled}(s) = \frac{f(s)}{\sqrt{\int_D \kappa(s)ds}} \]

**Centering**

\[ f_{centered}(s) = f(s) - \frac{\int_D f(s)\kappa(s)ds}{\int_D \kappa(s)ds} \]
Desired invariance in rotation
Motivation
Defining the distance function
The Riemannian distance function
Optimization over re-parametrization
Experimental results

Framework
Introducing q-maps
Invariance in rigid motion and global scaling
Invariance in re-parametrization

Desired invariance in rotation

Definition: Rotation Group

The rotation group is given by $SO(3)$. There we define a left group action of $SO(3)$ on $\mathcal{F}$, which means

$$SO(3) \times \mathcal{F} \rightarrow \mathcal{F}, (O, f) = Of \quad \text{for } O \in SO(3) \text{ and } f \in \mathcal{F}$$
Desired invariance in rotation

Theorem

The constructed metric is invariant to rotation.
Proof:
Let $O \in SO(3)$ and $f_1, f_2 \in \mathcal{F}$, then:

$$d(Of_1, Of_2)^2 = \int_D \|Of_1(s) - Of_2(s)\|^2 ds$$

$$= \int_D \|O\|^2_2 \|f_1(s) - f_2(s)\|^2 ds$$

$$= \int_D \lambda_{\text{max}}(O^T O) \|f_1(s) - f_2(s)\|^2 ds$$

$$= \int_D \|f_1(s) - f_2(s)\|^2 ds$$

$$= d(f_1, f_2)^2$$

This shows that $d(f_1, f_2)$ is invariant to rigid transformation and global scaling.
Invariance in re-parametrization
Desired invariance in re-parametrization

Definition

The set of all area preserving parametrizations is given by

\[ \Gamma = \{ \gamma : D \to D \text{ diffeomorphism} \} \]
Definition: Re-parametrization group

The re-parametrization group is given by $\Gamma$. We define the right group action of $\Gamma$ on $\mathcal{F}$ as

$$\mathcal{F} \times \Gamma \rightarrow \mathcal{F}, (f, \gamma) = (f \circ \gamma) \quad \text{for } f \in \mathcal{F} \text{ and } \gamma \in \Gamma$$
Definition: Re-parametrization group

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Especially the right group action for q-maps is given by

\[ \mathcal{F} \times \Gamma \to \mathcal{F}, (q, \gamma) = \sqrt{\det J_\gamma(q \circ \gamma)} \]
Now that we now how the re-parametrizations of q-maps look like, we can check the isometry condition.

**Theorem**

The re-parametrization group $\Gamma$ acts on $\mathcal{F}$ by isometries. That is, for any q-maps $q_1, q_2 \in \mathcal{F}$ and $\gamma \in \Gamma$ we have:

$$d((q_1, \gamma), (q_2, \gamma)) = d(q_1, q_2)$$
Proof:
Let $q_1, q_2 \in \mathcal{F}$ and $\gamma \in \Gamma$. Then:

$$d((q_1, \gamma), (q_2, \gamma))^2 = \int_D \|(q_1, \gamma) - (q_2, \gamma)\|^2 ds$$

$$= \int_D \|\sqrt{\det J_\gamma(q_1(\gamma(s))) - q_2(\gamma(s))}\|^2 ds$$

$$= \int_D \det J_\gamma \|q_1(\gamma(s)) - q_2(\gamma(s))\|^2 ds$$

$$= \int_D \|q_1(s) - q_2(s)\|^2 ds$$

$$= d(q_1, q_2)^2$$
So far:

We defined a distance function $d(q_1, q_2) = \left( \int_D ||q_1 - q_2||^2 \right)^{\frac{1}{2}}$ on q-maps that is invariant to

- re-parametrization
- global scaling
- rigid motion.
So far:
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  - re-parametrization
  - global scaling
  - rigid motion.

Now:
Define the Riemannian distance function to compute the minimal distance between two surfaces.
The Riemannian distance function

Definition: Riemannian distance

The Riemannian distance function between two surfaces $S_1$, $S_2$ represented by their q-maps $q_1$ and $q_2$, is given by

$$d([q_1], [q_2]) = \inf_{O \in SO(3), \gamma \in \Gamma} \| q_1 - (Oq_2, \gamma) \|$$

It is a method to **measure the differences** of the shape of two parameterized surfaces.
Quotient-space $S$

**Definition:**

The elements of the quotient-space $S \equiv \mathcal{F}\backslash(\Gamma \times SO(3))$ are given by

$$[q] = \{(Oq, \gamma)|O \in SO(3), \gamma \in \Gamma\}$$
Actions of $\Gamma$ and $SO(3)$ on $\mathcal{F}$ commute

**Theorem**

Actions of $\Gamma$ and $SO(3)$ on $\mathcal{F}$ commute, which means that

$$O \circ (q \circ \gamma) = (O \circ q) \circ \gamma$$
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**Consequence:**

We can search independently for the optimal rotation $O^*$ and optimal re-parametrization $\gamma^*$ and the geodesic between $q_1$ and $[q_2]$ will be given by

$$O^* \circ (q_2 \circ \gamma^*)$$
Optimization over rotation rotation

Algorithm:
Let \( \tilde{q}_2 \) denote \((q_2, \gamma)\). For a fixed \( \gamma \in \Gamma \) the best rotation \( O^* \) can be computed as follows:

1. Compute \( A = \int_D q_1(s)\tilde{q}_2(s)^T \, ds \), \( A \in \mathbb{R}^{3 \times 3} \).
2. Use singular value decomposition \( A = U\Sigma V^T \).
3. Optimal rotation is given by \( O^* = VU^T \).
Optimization over rotation
Now we have found the optimal rotation to compute the minimal distance between \( q_1 \) and \( q_2 \).

We still need to find a solution to

\[
\inf_{\gamma \in \Gamma} \| q_1 - (q_2, \gamma) \|
\]

This can be done by using the **method of gradients**.
Reminder: Method of gradients can be used to find a local(!) optimum of an optimization problem like

\[
\min_{x \in \mathbb{R}^d} f(x)
\]

Method: For each step \( k \) and step size \( \epsilon > 0 \) solve

\[
x^{k+1} = x^k - \epsilon \nabla f(x^k)
\]
Method of gradients

Here we want to optimize:

\[
\inf_{r \in [q_2]} E(r) = \inf_{r \in [q_2]} ||q_1 - r|| = \inf_{\gamma \in \Gamma} ||q_1 - (q_2, \gamma)||
\]

The approach for the method of gradients would then be:

\[
r^{k+1} = r^k - \epsilon v \quad \text{with} \quad v = \nabla E(r) = -\frac{q_1 - r}{||q_1 - r||}
\]

Unfortunately \(v\) is not necessarily in \(T_r([q_2])\) which means that it can happen that \(r^{k+1} \notin [q_2]\)
\[\Rightarrow\] we need to project \(v\) on \(T_r([q_2])\)
Method of gradients

**Approach:**
We try to solve
\[
\inf_{\gamma \in \Gamma} \| q_1 - (q_2, \gamma) \| = \inf_{\gamma \in \Gamma} \| q_1 - \Psi(\gamma) \| = \inf_{\gamma \in \Gamma} E(\gamma)
\]
instead, where \( \Psi : \Gamma \rightarrow [q_2], \Psi(\gamma) = (q_2, \gamma) \).
Approach:
We try to solve

\[
\inf_{\gamma \in \Gamma} ||q_1 - (q_2, \gamma)|| = \inf_{\gamma \in \Gamma} ||q_1 - \Psi(\gamma)|| = \inf_{\gamma \in \Gamma} E(\gamma)
\]

instead, where \(\Psi : \Gamma \rightarrow [q_2], \Psi(\gamma) = (q_2, \gamma)\).

Therefore we need:

- an orthonormal basis \(B\) of \(T_{\gamma_{id}}(\Gamma)\)
- the differential \(\Psi_*\) at an identity point \(\gamma_{id}(s) = s\)
- the full differential \(\frac{dE(\gamma)}{d\gamma} =: d\gamma\)
The orthonormal basis $\mathcal{B}$ of $T_{\gamma id}(\Gamma)$

The tangent space $T_{\gamma id}$ is given by

$$T_{\gamma id}(\Gamma) = \{ b : S^2 \rightarrow T(S^2) | b \text{ is a smooth VF and tangential to } S^2 \}$$

For the orthonormal basis $\mathcal{B}$ we choose basis elements $b_1, b_2, \ldots$ of $T_{\gamma id}(\Gamma)$ using spherical harmonics. These are nothing but the Eigenfunctions of the Laplace-Beltrami-Operator on a sphere.
Basis elements of $T_{\gamma_{id}}(\Gamma)$

Figure: Four examples of basis elements of $T_{\gamma_{id}}(\Gamma)$
The differential $\Psi_\star$ at $\gamma_{id}$

**Definition**

Let $b$ be an element of $T_{\gamma_{id}}(\Gamma)$ and let $r$ be the current estimation of the nearest element in $[q_2]$ to $q_1$. Then the differential of $\Psi_{\gamma_{id}}$ at $b$ is given by

$$\Psi_{\star,\gamma_{id}}(b) = \nabla r_j \cdot b, \quad j = 1, 2, 3$$
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Method of gradients
Algorithm

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Now we can define the full gradient of $E(\gamma)$ by

$$d\gamma := \frac{dE}{d\gamma} = \sum_i \langle v, \Psi^*(b_i) \rangle b_i \in T_{\gamma id}(\Gamma)$$

and update our current estimate $r$ by computing

$$r^{k+1} = r^k \circ (\gamma id - \epsilon d\gamma)$$
Schematical algorithm

\[ r^k := r^k \circ \gamma_{id} \]

Updating \( r^k \)

Computing \( d\gamma \) by using gradient approach

\[ r^{k+1} := r^k \circ (\gamma_{id} + \varepsilon \, d\gamma) \]

Break condition:
Repeat until \( ||d\gamma|| \) becomes sufficiently small
The algorithm

Let $r \in [q_2]$ be the current estimate of the element nearest to $q_1$. Also, let $\{b_1, i = 1, \ldots k\}$ be the orthonormal basis for $T_{\gamma_{id}}[\Gamma]$ for some $k$. The choice of $k$ is dictated by the computational resources versus the desired accuracy.

1. Compute the tangent vector $v = \frac{q_1 - r}{\|q_1 - r\|}$
2. Compute $\Phi_\ast(b_i)$ for each $b_i$
3. Estimate the gradient vector field $d\gamma$
4. Update the parametrization: $\gamma_{\text{new}} = \gamma_{id} + \epsilon d\gamma$
5. Re-parameterize the second surface $r$ according to $r \circ \gamma_{\text{new}}$. Return to step 1 until $\|d\gamma\|$ becomes sufficiently small.
Now we have:

- A method to compare and match different shapes in $\mathbb{R}^3$
- A new framework to compute geodesics between parameterized surfaces
- Geodesics are invariant to rigid motion, scaling and re-parametrization
Matching shapes
Comparing shapes
Thanks for your attention!