Emanuel Laude

Department of Informatics - Technische Universität München

Abstract

The study of the Laplace operator and its corresponding eigenvalue problem is crucial to understand the foundations of 3D shape analysis. For that reason the most important mathematical properties of the Laplace operator in Euclidean spaces, its eigenvalues and eigenfunctions are summarized and explained in this report. The basic definitions and concepts of infinite dimensional function spaces, that are required are introduced beforehand. In order to motivate the study of the Laplace eigenvalue problem, the connection between the Laplace eigenvalues and the geometry of the domain, to which the operator is applied, is demonstrated by presenting Weyl's law. Furthermore it is shown how the Fourier transform can be derived from the consequences of the spectral theorem and orthogonal basis functions of the $L^2$ space. Finally some applications of the Laplace eigenfunctions in 3D shape analysis are demonstrated briefly.

1 Introduction: Why eigenfunctions of the Laplace operator?

The Laplace operator occurs in a couple of partial differential equations that describe basic physical phenomena such as the propagation of waves or diffusion processes. One of the standard recipes to tackle those equations is the so called method of separation of variables or Fourier method which involves the solution of the Laplace eigenvalue problem. At first sight there is no reason why one should expect those equations to be related to the geometry of surfaces or shapes. But it turns out that there is in fact a strong connection between the Laplace eigenfunctions, the partial differential equations and finally the geometrical properties of the domain $\Omega$ to which the Laplace operator is applied. This gives rise to the following central questions known as the direct and the inverse problem, that finally motivate the study of the Laplace operator and its corresponding eigenvalue problem:

Direct problem: If the shape of a domain is known, what can be deduced of its Laplace eigenvalues?

Inverse problem: If the Laplace eigenvalues of a domain are known, what can be deduced of its geometry?

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be a euclidean, connected domain and $\varphi : \Omega \to \mathbb{C}$ be a function. Then the Laplace operator is defined as follows:

$$\Delta \varphi = -\text{div}\nabla \varphi = -\sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial x_i^2} \quad (1)$$

Remark 1. Three-dimensional shapes can be represented by their surfaces, so called two-dimensional Riemannian manifolds. Usually those surfaces have no boundary. In order to apply the Laplace operator to Riemannian manifolds it has to be generalized. It is then called Laplace-Beltrami operator. In this report we focus on the ordinary Laplace operator which is applied to planar euclidean domains. However, most properties that are derived also hold for the Laplace-Beltrami operator.

Obviously the Laplace operator is linear due to the linearity of the sum and the partial derivatives. More formally: Let $\Omega \subset \mathbb{R}^n$ and $f, g : \Omega \to \mathbb{C}$ be twice differentiable functions and $\alpha, \beta$ scalars. Then:

$$\Delta(\alpha f + \beta g)(x) = \alpha \Delta f(x) + \beta \Delta g(x) \quad (2)$$

Due to the fact that the Laplace operator is not a matrix but a linear operator the concept of eigenvectors has to be generalized to eigenfunctions:
First of all the concept of eigenvalues and eigenvectors remains valid if the underlying space is an infinite dimensional function space e.g. the $L^2$ space.

A matrix $A \in \mathbb{C}^{m \times n}$ can be generalized to a linear operator $T : L^2(\Omega) \rightarrow L^2(\Omega)$.

Eigenvectors are generalized to eigenfunctions.

The actual eigenvector problem becomes $T\varphi = \lambda\varphi$ where $\varphi : \Omega \rightarrow \mathbb{C}$ is a function.

Components of vectors are no longer indexed by integers of the set $\{1, \ldots, n\}$ but real values. Indices correspond to the arguments of functions.

Symmetric or hermitian matrices are generalized to self-adjoint linear operators.

**Definition 2** ($L^2$ space). Let $\Omega \subset \mathbb{R}^n$. Then

$$L^2(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{C} \mid \int_{\Omega} |f(x)|^2 \, dx < \infty \right\}$$

with corresponding inner product

$$\langle f, g \rangle = \int_{\Omega} f(x) \overline{g(x)} \, dx, \quad f, g \in L^2(\Omega)$$

is an infinite dimensional vector space called $L^2$ space.

## 2 The Laplace operator on Intervals

The following Example is taken from [1].

Let the interval $\Omega = [0, l]$ be the domain of the function $\varphi : \Omega \rightarrow \mathbb{R}$. Solve the eigenvalue problem

$$\Delta \varphi = \lambda \varphi$$

subject to the so called Dirichlet boundary conditions: $\varphi(0) = \varphi(l) = 0$. Then the eigenfunctions are

$$\varphi_k(x) = \sin \left( \frac{k\pi}{l} x \right) \text{ for } k \geq 1$$

with eigenvalues $\lambda_k = \left( \frac{k\pi}{l} \right)^2$ for $k \geq 1$.

The eigenfunctions $\varphi_k$ of the example above can be considered as the fundamental tone and the overtones of a vibrating guitar string of length $l$. The eigenfunction $\varphi_1$ corresponds to the fundamental tone. The Dirichlet boundary conditions ensure that no vibration occurs at the ends of the string since they are attached to the instrument.

The solution of the above eigenvalue problem subject to the so called von Neumann boundary conditions $\varphi'(0) = \varphi'(l) = 0$:

The eigenfunctions are

$$\varphi_k(x) = \cos \left( \frac{k\pi}{l} x \right) \text{ for } k \geq 0$$

with corresponding eigenvalues $\lambda_k = \left( \frac{k\pi}{l} \right)^2$ for $k \geq 0$.

## 3 Solution of the wave equation

The following Example is adapted from [1], [2] and [4].

Whereas the eigenvalue problem explained above describes the overtones of the vibrating guitar string separately, the wave equation

$$\frac{\partial^2}{\partial t^2} u(x, t) = -\Delta u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t)$$

is given by

$$\frac{\partial^2}{\partial t^2} u(x, t) = -\Delta u(x, t)$$

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models the actual vibration of the guitar string and thereby describes how the different overtones are combined into one characteristic tone: For each point $t$ in time $u(x, t)$ gives the height of the guitar string above the point $x \in [0, l]$.

In order to uniquely determine the vibration process also the initial shape of the string $f(x) = u(x, 0)$ and the initial velocity $g(x) = \partial_t u(x, 0)$ have to be known as well as the boundary condition

$$u(0, t) = u(l, t) = 0$$

which encodes the fact that the ends of the string are attached to the instrument. Then the overall problem becomes:

$$
\begin{align*}
\frac{\partial^2}{\partial t^2} u(x, t) &= \frac{\partial^2}{\partial x^2} u(x, t) \\
u(0, t) &= u(l, t) = 0 \\
f(x) &= u(x, 0) \\
g(x) &= \partial_t u(x, 0)
\end{align*}
$$

By applying the method of separation of variables $u(x, t)$ is decomposed into $u(x, t) = \varphi(x) \alpha(t)$.

Consequently equation 6 can be written as

$$\varphi(x) \alpha''(t) = \varphi''(x) \alpha(t)$$

or equivalently

$$\frac{\alpha''(t)}{\alpha(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda$$

As the LHS of equation 9 only depends on $t$ and the RHS only on $x$ both sides of the equation have to be equal to a constant $\lambda$:

$$\frac{\alpha''(t)}{\alpha(t)} = \frac{\varphi''(x)}{\varphi(x)} = \lambda$$

Including the boundary condition this gives rise to the following two eigenvalue problems:

The first eigenvalue problem includes boundary condition 7 since the variable $x$ only occurs in $\varphi(x)$. It is equal to the Dirichlet eigenvalue problem 3:

$$
\begin{align*}
\varphi''(x) &= \lambda \varphi(x) \\
\varphi(0) &= \varphi(l) = 0
\end{align*}
$$

The second eigenvalue problem is:

$$\alpha''(t) = \lambda \alpha(t)$$

The solution of the first eigenvalue problem is equal to 4. The solution of the joint eigenvalue problem with common $\lambda_k = \left(\frac{k\pi}{l}\right)^2$ then is:

$$
\begin{align*}
\varphi_k(x) &= \sin \left(\frac{k\pi}{l} x\right) \\
\alpha_k(t) &= a_k \cos \left(\frac{k\pi}{l} t\right) + b_k \sin \left(\frac{k\pi}{l} t\right)
\end{align*}
$$

Every $u_k = \alpha_k \varphi_k$ satisfies the original problem consisting of the partial differential equation and the boundary condition. To make sure that the final solution also satisfies the initial conditions one has to apply the principle of superposition: If the original problem is linear and both $u_1(x, t)$ and $u_2(x, t)$ satisfy the original problem then also $\alpha_1 u_1(x, t) + \beta u_2(x, t)$ satisfies the problem. Assuming that this holds as well for infinite linear combinations, a general solution of the problem then has the form:

$$u(x, t) = \sum_{k=1}^{\infty} \alpha_k(t) \varphi_k(x)$$
The coefficients $a_k$ and $b_k$ are determined as follows: For $t = 0$ it has to hold
\[ u(x, 0) = \sum_{k=1}^{\infty} \alpha_k(0) \varphi_k(x) = \sum_{k=1}^{\infty} a_k \varphi_k(x) = f(x) \] (15)
and
\[ \partial_t u(x, 0) = \sum_{k=1}^{\infty} \partial_t \alpha_k(0) \varphi_k(x) = \sum_{k=1}^{\infty} b_k \frac{k\pi}{l} \varphi_k(x) = g(x) \] (16)
The two problems above are similar in the sense that one can obtain their solutions by solving the following equation for $c_n$ where $h(x)$ and $\varphi_n(x)$ are given:
\[ h(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x) \] (17)
The sequence of functions \( \{ \varphi_n(x) = \sin \left( \frac{n\pi}{l} x \right) \}_{n=1}^{\infty} \) is orthogonal in the sense that
\[
\int_0^l \varphi_m(x) \varphi_n(x) \, dx = \int_0^l \sin \left( \frac{m\pi}{l} x \right) \sin \left( \frac{n\pi}{l} x \right) \, dx = \begin{cases} \frac{l}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}
\]
If both sides of equation (17) are multiplied by $\varphi_k(x)$ and integrated over $[0, l]$ one obtains:
\[
\int_0^l h(x) \varphi_k(x) \, dx = \int_0^l \varphi_k(x) \sum_{n=1}^{\infty} c_n \varphi_n(x) \, dx = \sum_{n=1}^{\infty} c_n \int_0^l \varphi_k(x) \varphi_n(x) \, dx = \frac{c_k l}{2}
\] or equivalently
\[ c_k = \frac{2}{l} \int_0^l h(x) \varphi_k(x) \, dx \] (18)
Finally one obtains:
\[ a_k = \frac{2}{l} \int_0^l f(x) \varphi_k(x) \, dx = \frac{2}{l} \langle f(x), \varphi_k(x) \rangle \] (19)
and
\[ b_k = \frac{2}{k\pi} \int_0^l g(x) \varphi_k(x) \, dx = \frac{2}{k\pi} \langle g(x), \varphi_k(x) \rangle \] (20)

4 The spectrum of the Laplace operator

**Definition 3** (Self-adjoint operator). A self-adjoint operator $T$ on a vector space $V$ with inner product $\langle \cdot, \cdot \rangle$ is an operator that is its own adjoint $T = T^*$ such that:
\[ \langle Tv, w \rangle = \langle v, T^* w \rangle = \langle v, Tw \rangle \]

**Remark 2.** If $V$ is finite dimensional and real, this is equivalent to the condition that $T$ is a symmetric matrix:
\[ \langle Tv, w \rangle = (Tv)^\top w = v^\top T^\top w = \langle v, T^\top w \rangle = \langle v, Tw \rangle \]

In case of one-dimensional domains such as intervals the Laplace operator is self-adjoint if functions $f$ satisfy $f(a) = f(b)$:

**Theorem 1.** Let $\Omega = [a, b] \subset \mathbb{R}$ and $f, g : \Omega \to \mathbb{C}$ be twice differentiable functions with $f(a) = f(b)$ and $g(a) = g(b)$. Then the Laplace operator is self-adjoint.
Proof. Let $\Omega = [a, b]$. By integrating by parts twice one obtains:

$$\langle \Delta f, g \rangle = \int_{\Omega} \Delta f(x) \overline{g(x)} \, dx$$

$$= \int_{\Omega} \frac{d^2 f(x)}{dx^2} \overline{g(x)} \, dx$$

$$= \left[ \frac{d}{dx} f(x) \overline{g(x)} - f(x) \frac{d}{dx} \overline{g(x)} \right]_{a}^{b} + \int_{\Omega} f(x) \frac{d^2 g(x)}{dx^2} \, dx$$

$$= \langle f, \Delta g \rangle$$

\[ \square \]

**Corollary 1.** Let $\Omega = [-\pi, \pi] \subset \mathbb{R}$ and $f, g : \Omega \rightarrow \mathbb{C}$ be twice differentiable $2\pi$-periodic functions. Then the Laplace operator is self-adjoint.

**Proof.** Let $f$ be twice differentiable and $2\pi$-periodic. I.e. $f(x + 2\pi) = f(x)$. This implies that $f(-\pi) = f(\pi)$

By theorem 1 it follows that the Laplace operator is self-adjoint. \[ \square \]

In case of $n$-dimensional domains functions $f$ have to vanish on the boundary to guarantee the self-adjointness of the Laplace operator.

**Theorem 2.** Let $\Omega \subset \mathbb{R}^n$ and $f, g : \Omega \rightarrow \mathbb{C}$ be twice differentiable functions with compact support (i.e. they vanish on the boundary). Then the Laplace operator is self-adjoint.

**Proof.** Let $\Omega \subset \mathbb{R}^n$. By integrating by parts twice we obtain:

$$\langle \Delta f, g \rangle = \int_{\Omega} \Delta f(x) \overline{g(x)} \, dx$$

$$= -\sum_{i=1}^{n} \int_{\Omega} \frac{\partial^2 f(x)}{\partial x_i^2} \overline{g(x)} \, dx$$

$$= -\sum_{i=1}^{n} \left( \int_{\partial \Omega} \left( \frac{\partial f(x)}{\partial x_i} \overline{g(x)} - f(x) \frac{\partial \overline{g(x)}}{\partial x_i} \right) \nu_i \, dx + \int_{\Omega} f(x) \frac{\partial^2 g(x)}{\partial x_i^2} \, dx \right)$$

$$= -\sum_{i=1}^{n} \int_{\Omega} f(x) \frac{\partial^2 g(x)}{\partial x_i^2} \, dx$$

$$= \langle f, \Delta g \rangle$$

where $\nu$ is the outward unit surface normal to $\partial \Omega$. $\nu_i$ is its $i$-th component, and $i$ ranges from 1 to $n$. \[ \square \]

**Definition 4.** A compact operator is a linear operator $T$ from a Banach space $X$ to another Banach space $Y$, such that the image under $T$ of any bounded subset of $X$ is a relatively compact subset of $Y$. A relatively compact subspace (or relatively compact subset) $N$ of a topological space $M$ is a subset whose closure is compact.

**Remark 3.** $L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$ is a Banach space.

**Theorem 3.** The Laplace operator on $L^2(\Omega)$ is compact.

Due to the self-adjointness and the compactness of the Laplace operator the Spectral Theorem can be applied to it.

**Theorem 4** (Spectral Theorem). For every compact self-adjoint operator $T$ on $L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$, there exists a sequence of eigenvalues

$$\lambda_1, \lambda_2, \lambda_3, \ldots \subset \mathbb{R}$$

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and an orthonormal basis of $L^2(\Omega)$

$$\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots$$

consisting of eigenfunctions of $T$ such that:

$$T\varphi_k = \lambda_k \varphi_k$$

Furthermore one can show that the Laplace operator is positive which has some important consequences for the eigenvalues and eigenfunctions of the Laplace operator:

**Theorem 5. The Laplace operator is positive.**

*Proof.* Let $\Omega \subset \mathbb{R}^n$. Furthermore let $f : \Omega \to \mathbb{C}$ be a twice differentiable function with compact support.

By integrating by parts we obtain:

$$\langle \Delta f, f \rangle = \int_{\Omega} \Delta f(x) f(x) \, dx$$

$$= - \sum_{i=1}^{n} \int_{\Omega} \frac{\partial^2 f(x)}{\partial x_i^2} f(x) \, dx$$

$$= - \sum_{i=1}^{n} \left( \int_{\partial \Omega} \frac{\partial f(x)}{\partial x_i} f(x) \nu_i \, dx - \int_{\Omega} \frac{\partial f(x)}{\partial x_i} \frac{\partial f(x)}{\partial x_i} \, dx \right)$$

$$= \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial f(x)}{\partial x_i} \right|^2 \, dx \geq 0$$

where $\nu$ is the outward unit surface normal to $\partial \Omega$. $\nu_i$ is its $i$-th component, and $i$ ranges from 1 to $n$. \qed

As mentioned above the theorem has some effects on the eigenfunctions and eigenvalues: Let $\varphi$ be an eigenfunction of the Laplace operator. It holds that

$$\lambda \langle \varphi, \varphi \rangle = \langle \lambda \varphi, \varphi \rangle = \langle \Delta \varphi, \varphi \rangle \geq 0$$

Since $\varphi \neq 0$ it holds that $\langle \varphi, \varphi \rangle \neq 0$. This implies that

$$\lambda = \frac{\langle \Delta \varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle} \geq 0$$

Consequently $\lambda = 0$ iff

$$\langle \Delta \varphi, \varphi \rangle = \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial \varphi(x)}{\partial x_i} \right|^2 \, dx = 0$$

or equivalently $\lambda = 0$ iff $\varphi$ is constant (in order to satisfy the above equation all partial derivatives have to be zero which is equivalent to $\varphi(x) = c, \forall x$).

What does this finally imply?

- In case we require the Dirichlet boundary conditions to hold the corresponding eigenfunction $\varphi(x) = c$ of eigenvalue $\lambda = 0$ has to be the zero function: $\varphi(x) = c = 0 = 0(x), \forall x$

- This is not permitted since eigenfunctions have to be non-zero ($\varphi \neq 0$). Consequently there is no eigenvalue $\lambda = 0$.

- In case of three-dimensional shapes represented as two-dimensional compact Riemannian manifolds without boundary ($\Rightarrow$ no boundary conditions) the corresponding eigenfunction of $\lambda = 0$ is the constant function $\varphi(x) = c$ with

$$c = \sqrt{\frac{1}{\text{Vol}(\Omega)}}$$

since $\langle c, c \rangle = c^2 \cdot \text{Vol}(\Omega) = 1$.  

6
5 Relation between the Laplace eigenvalues and geometry

As pointed out in the beginning it can be shown that the Laplace operator carries some information about the geometry of the domain to which it is applied. Weyl’s Law relates the eigenvalues of the Laplace operator to the volume of the domain: Knowing the eigenvalues and therefore their growth rate one can infer the volume of the domain in case of three-dimensional shapes or the area of the domain in case of planar Euclidean shapes. Moreover this can be considered as an instance of the inverse problem:

The following theorem is taken from [1].

**Theorem 6.** Let \(0 < \lambda_1 \leq \lambda_2 \leq \ldots\) be the Dirichlet eigenvalues of the Laplace Operator on a bounded domain \(\Omega \subset \mathbb{R}^n\). Then

\[
N(\lambda) := \#\{\lambda_k \mid \lambda_k \leq \lambda\} \sim \frac{\omega_n}{(2\pi)^n} Vol(\Omega) \lambda^{n/2}
\]

as \(\lambda \to \infty\). \(\omega_n\) is the volume of the unit ball in \(\mathbb{R}^n\). It further holds that:

\[
N(\lambda_j) = \max\{k \mid \lambda_k \leq \lambda_j\} = j
\]

Solving for \(\lambda_j\) yields:

\[
\lambda_j \sim \frac{(2\pi)^2}{(\omega_n Vol(\Omega))^{2/n}} j^{2/n}
\]

The following proof is adapted from [1]. Weyl’s law is proven for the special case that \(\Omega\) is an interval, that is \(n = 1\):

Let \(0 < \lambda_1 \leq \lambda_2 \leq \ldots\) be the Dirichlet eigenvalues of the Laplace Operator on \(\Omega = [0,l] \subset \mathbb{R} = \mathbb{R}^1\). Then

\[
N(\lambda) = \#\{\lambda_k \mid \lambda_k \leq \lambda\} = \max\{k \mid \lambda_k \leq \lambda\} = \max\left\{k \left| \frac{k^2 \pi^2}{l^2} \leq \lambda \right. \right\} \sim \frac{l}{\pi \sqrt{\lambda}} = \frac{\omega_n}{(2\pi)^n} Vol(\Omega) \lambda^{n/2}
\]

In case of three-dimensional shapes \(n = 2\) and therefore \(\omega_n = \pi\). Consequently we obtain:

\[
\lambda_j \sim \frac{(2\pi)^2}{(\omega_n Vol(\Omega))^{2/n}} j^{2/n} = \frac{4\pi}{Vol(\Omega)} j
\]

This implies that the eigenvalues grow linearly in case of three-dimensional shapes and quadratically in case of intervals.

6 The Fourier Transform

The idea behind the Fourier Transform or more concretely the Fourier series is to express any function that is \(2\pi\)-periodic and square integrable as an infinite sum of certain basis functions namely \(e^{ikx}\). This concept is closely related to orthonormal bases of the infinite dimensional \(L^2\) space.

**Theorem 7** (Fourier basis).

\[
\{e_k = \frac{1}{\sqrt{2\pi}} e^{ikx} \mid k \in \mathbb{N} \cup \{0\}\}
\]

forms an orthonormal basis of \(\{f \in L^2([\pi, \pi]) \mid f(x + 2\pi) = f(x)\}\).
Obviously the functions \( \frac{1}{\sqrt{2\pi}} e^{ikx} \) are eigenfunctions of the one-dimensional Laplace operator. Moreover the Laplace operator defined on

\[
\{ f \in L^2([-\pi, \pi]) \mid f(x + 2\pi) = f(x) \}
\]

is self-adjoint by corollary \ref{corollary:l-adj}. Therefore the Spectral theorem can be used to show that \( \{ e_k = \frac{1}{\sqrt{2\pi}} e^{ikx} \mid k \in \mathbb{N} \cup \{0\} \} \) is indeed a basis of \( \{ f \in L^2([-\pi, \pi]) \mid f(x + 2\pi) = f(x) \} \). Orthogonality follows from:

\[
\langle e^{inx}, e^{imx} \rangle = \int_{-\pi}^{\pi} e^{i(n-m)x} \, dx = \begin{cases} 2\pi & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}
\]

The following lemma is adapted from \cite{3}

**Lemma 1.** Let \( \{ \varphi_k \mid k \in \mathbb{N} \cup \{0\} \} \) be an orthonormal set in \( L^2(\Omega) \). \( \{ \varphi_k \mid k \in \mathbb{N} \cup \{0\} \} \) is an orthonormal basis if and only if for all \( x \in L^2(\Omega) \)

\[
x = \sum_{k=0}^{\infty} \langle x, \varphi_k \rangle \varphi_k
\]

**Proof.** Let \( x \in L^2(\Omega) \). Since \( \{ \varphi_k \mid k \in \mathbb{N} \cup \{0\} \} \) is a basis of \( L^2(\Omega) \) there exist scalars \( \alpha_k \) such that \( x = \sum_{k=0}^{\infty} \alpha_k \varphi_k \). Due to the continuity and the linearity of the inner product it then holds for arbitrary \( j \in \mathbb{N} \cup \{0\} \) that:

\[
\langle x, \varphi_j \rangle = \left( \sum_{k=0}^{\infty} \alpha_k \varphi_k, \varphi_j \right) = \sum_{k=0}^{\infty} \alpha_k \langle \varphi_k, \varphi_j \rangle = \alpha_j \langle \varphi_j, \varphi_j \rangle = \alpha_j
\]

Conversely suppose that there exist scalars \( \alpha_k \neq \langle x, \varphi_k \rangle \) such that \( x = \sum_{k=0}^{\infty} \alpha_k \varphi_k \). With exactly the same reasoning as above one can show that \( \alpha_k = \langle x, \varphi_k \rangle \) which shows that the expansion of \( x \) is unique and consequently \( \{ \varphi_k \mid k \in \mathbb{N} \cup \{0\} \} \) is a basis (and by assumption an orthonormal basis).

Let \( f \in L^2([-\pi, \pi]) \) be \( 2\pi \)-periodic. Then by theorem \ref{theorem:pl-decomposition} there exist scalars \( \alpha_k \) such that

\[
f = \sum_{k=0}^{\infty} \alpha_k \varphi_k(x)
\]

with \( \varphi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} \).

By lemma \ref{lemma:orthonormality} it further holds that

\[
\alpha_k = \langle f, \varphi_k \rangle = \int_{-\pi}^{\pi} f(x) \varphi_k(x) \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx =: \hat{f}(k)
\]

such that one can rewrite equation \ref{equation:pl-decomposition} as

\[
f = \sum_{k=0}^{\infty} \hat{f}(k) \frac{1}{\sqrt{2\pi}} e^{ikx}
\]

Moreover it can be proven, that the convergence rate of the expansion \( f = \sum_{k=0}^{\infty} \hat{f}(k) \frac{1}{\sqrt{2\pi}} e^{ikx} \) depends on smoothness of the function \( f \): The smoother the faster the convergence.

The Fourier series can be generalized to compact Riemannian manifolds:

**Theorem 8** (Sturm-Liouville’s decomposition). Given a compact Riemannian manifold \((M, g)\) there is an orthonormal basis \( \varphi_1, \ldots, \varphi_j, \ldots \) of eigenfunctions of the Laplace-Beltrami Operator with respective eigenvalues \( 0 \leq \lambda_1 \leq \ldots \leq \lambda_j \leq \ldots \) such that any functions \( \phi \in L^2(M) \) can be written as a convergent series in \( L^2(M) \)

\[
\phi = \sum_{j=1}^{\infty} a_j \varphi_j
\]

for some coefficients \( a_j \in \mathbb{R} \).
The theorem is taken from [1].

Sturm-Liouville’s decomposition can be applied in 3D shape analysis as follows: One can describe a surface $M \subset \mathbb{R}^3$ by functions such as the normal vectors. That is for every point $x \in M$ one can associate the normal vector $\mathbf{n}$ at $x$:

$$\mathbf{n}(x) = (n_1(x), \ n_2(x), \ n_3(x))$$

By the Sturm-Lioville’s decomposition one can rewrite the functions $n_j : M \to \mathbb{R}$ as infinite series $n_j = \sum_{i=1}^{\infty} a_i^{(j)} \varphi_i$. One may truncate the series and work with an approximation of the manifold described by the function

$$\mathbf{n}(x) = \left( \sum_{i=1}^{N} a_i^{(1)} \varphi_i(x), \ \sum_{i=1}^{N} a_i^{(2)} \varphi_i(x), \ \sum_{i=1}^{N} a_i^{(3)} \varphi_i(x) \right)$$

The example above is taken from [1].

7 Further examples of eigenfunctions of the Laplace operator

7.1 Eigenfunctions of the rectangle

Let $\Omega = [0, l] \times [0, m]$ be a rectangular domain. Solution of the corresponding eigenvalue problem subject to the Dirichlet boundary conditions:

Eigenfunctions are:

$$\varphi_{jk} = \sin \left( \frac{j\pi}{l} x \right) \sin \left( \frac{k\pi}{m} x \right)$$

with corresponding eigenvalues:

$$\lambda_{jk} = \left( \frac{j\pi}{l} \right)^2 + \left( \frac{k\pi}{m} \right)^2$$
7.2 Laplace eigenfunctions of the Disc (Vibrating drumhead)

Let $\Omega = \{ x \in \mathbb{R}^2 \mid |x| \leq 1 \}$ be a circular domain. Solution of the corresponding eigenvalue problem (in polar coordinates $r, \theta$):

Eigenfunctions are:

$$\varphi_k^\lambda(r, \theta) = \phi_k(\theta) J_k(\sqrt{\lambda} r)$$ (26)

where $\phi_k(\theta) = a_k \cos(k\theta) + b_k \sin(k\theta)$ and

$$J_k(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(k + l)!} \left( \frac{x}{2} \right)^{k+2l}$$ (27)

is the $k$-th Bessel function.
7.3 Outlook: Eigenfunctions of compact Riemannian manifolds

As stated in the beginning the Laplace operator can be generalized to compact Riemannian manifolds which can represent three-dimensional shapes by their surfaces:
Figure 5: The first five Laplace eigenfunctions of the cat including the constant eigenfunction corresponding to eigenvalue $\lambda = 0$. Color encodes amplitude of the eigenfunction. (Red: high value, blue: small value)

\section{Conclusion}

As stated in the beginning the Laplace operator and its eigenfunctions carry some information of the geometry of the domain to which the operator is applied. Weyl’s law demonstrates that there is in fact a mathematical relation between the volume of the domain and the growth rate of the eigenvalues. This result might not seem very strong but using other techniques such as the heat kernel [5] one can draw even more powerful geometrical secrets from the eigenfunctions. Most of the properties explained in this report are derived for the planar euclidean Laplace operator. However, it is possible to generalize them to Riemannian manifolds. Finally the idea of hitting a guitar string, recording its eigenfrequencies and deducing some geometrical properties can be transferred to compact Riemannian manifolds by using the Laplace-Beltrami operator.
References


