Heat Kernel Signature

Recent Advances in the Analysis of 3D Shapes

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History of Point Signatures
What is a Point Signature
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History

Matching of non rigid shapes
From 2D Image Processing

1. Defining a set of feature points
2. Find a “good” matching (RANSAC)
3. Verify the transformation
Point Signatures (1)

Point Signature from 1996

Global Point Signature (GPS) 2007
Heat Kernel Signature (HKS) 2009
Wave Kernel Signature (WKS) 2011
Point Signatures (2)

Point Signature from 1996

**Global Point Signature (GPS) 2007**

Heat Kernel Signature (HKS) 2009

Wave Kernel Signature (WKS) 2011

Given an ordering of the eigenvalues of the Laplacian \( (\Delta \phi = \lambda \phi) \)

\[
0 = \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots
\]

The Global Point Signature is defined as

\[
GPS(p) = \left( \frac{1}{\sqrt{\lambda_1}} \phi_1(p), \frac{1}{\sqrt{\lambda_2}} \phi_2(p), \frac{1}{\sqrt{\lambda_3}} \phi_3(p), \ldots \right)
\]
Eigenfunctions \((\Delta \phi = \lambda \phi)\)
A good Point Signature

Invariant under isometric deformation

Informative

Stable under noise

Multi scale

Scale invariant
Review

Scalar Product:

\[ \langle f, g \rangle := \int_M f(x)g(x)dx \]

Laplace Beltrami Operator

\[ \Delta f = \text{div} \nabla f \]

Eigenvalues and Eigenfunctions

\[ \Delta \phi_i = \lambda_i \phi_i \]
\[ \langle \phi_i, \phi_j \rangle = 0, \ i \neq j \]
\[ \sum_{i=0}^{\infty} \langle f, \phi_i \rangle \phi_i = f \]

Note: The eigenvalues are negative
Heat Equation

\[ \Delta_M u(x, t) = \frac{\partial u(x, t)}{\partial t} \]

Describes heat diffusion on a Manifold \( M \)

Initial Value: \( u(x, 0) = f(x) \)

A function \( k_t \) that solves the heat equation by

\[ u(x, t) = \int_M k_t(x, y)f(y)dy \]

is called heat kernel (fundamental solution)
Solving the heat equation (1)

Decompose the function $u$ by its eigenfunctions

$$u(x, t) = \sum_{i=0}^{\infty} \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x)$$
Solving the heat equation (1)

Solving the right side of the heat equation

\[ u(x, t) = \sum_{i=0}^{\infty} \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x) \]

\[ \frac{\partial}{\partial t} u(x, t) = \sum_{i=0}^{\infty} \frac{\partial}{\partial t} \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x) \]
Solving the heat equation (1)

Solving the left side of the heat equation

\[
\begin{align*}
\Delta u(x, t) &= \sum_{i=0}^{\infty} \Delta \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x) = \sum_{i=0}^{\infty} \langle u(t, \cdot), \phi_i(\cdot) \rangle \lambda_i \phi_i(x) \\
\frac{\partial}{\partial t} u(x, t) &= \sum_{i=0}^{\infty} \frac{\partial}{\partial t} \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x) \\
u(x, t) &= \sum_{i=0}^{\infty} \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x)
\end{align*}
\]
Solving the heat equation (1)

Compare the coefficients of the sum

\[ u(x, t) = \sum_{i=0}^{\infty} \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x) \]

\[ \frac{\partial}{\partial t} u(x, t) = \sum_{i=0}^{\infty} \frac{\partial}{\partial t} \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x) \]

\[ \Delta u(x, t) = \sum_{i=0}^{\infty} \Delta \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x) = \sum_{i=0}^{\infty} \langle u(t, \cdot), \phi_i(\cdot) \rangle \lambda_i \phi_i(x) \]

\[ \frac{\partial}{\partial t} \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x) = \langle u(t, \cdot), \phi_i(\cdot) \rangle \lambda_i \phi_i(x) \ \forall i \]
Solving the heat equation (1)

Solve the differential equation

\[ u(x, t) = \sum_{i=0}^{\infty} \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x) \]

\[ \frac{\partial}{\partial t} u(x, t) = \sum_{i=0}^{\infty} \frac{\partial}{\partial t} \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x) \]

\[ \Delta u(x, t) = \sum_{i=0}^{\infty} \Delta \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x) = \sum_{i=0}^{\infty} \langle u(t, \cdot), \phi_i(\cdot) \rangle \lambda_i \phi_i(x) \]

\[ \frac{\partial}{\partial t} \langle u(t, \cdot), \phi_i(\cdot) \rangle \phi_i(x) = \langle u(t, \cdot), \phi_i(\cdot) \rangle \lambda_i \phi_i(x) \quad \forall i \]

\[ \langle u(t, \cdot), \phi_i(\cdot) \rangle = d_i e^{\lambda_i t} \]
Solving the heat equation (2)

Use solution from before

\[ u(x, t) = \sum_{i=0}^{\infty} d_i e^{\lambda_i t} \phi_i(x) \]
Solving the heat equation (2)

Set initial value

\[ u(x, t) = \sum_{i=0}^{\infty} d_i e^{\lambda_i t} \phi_i(x) \]

\[ \lim_{t \to 0} u(x, t) = f \]
Solving the heat equation (2)

Set value for unknown $d_i$

$$u(x, t) = \sum_{i=0}^{\infty} d_i e^{\lambda_i t} \phi_i(x)$$

$$\lim_{t \to 0} u(x, t) = f$$

$$\sum_{i=0}^{\infty} d_i \phi_i(x) = f$$

$$d_i = \langle f, \phi_i \rangle$$
Solving the heat equation (3)

Use solution from ODE

\[ u(x, t) = \sum_{i=0}^{\infty} \langle f, \phi_i \rangle e^{\lambda_i t} \phi_i(x) \]
Solving the heat equation (3)

Use definition of scalar product

\[ u(x, t) = \sum_{i=0}^{\infty} \langle f, \phi_i \rangle e^{\lambda_i t} \phi_i(x) \]

\[ = \sum_{i=0}^{\infty} \left( \int_{M} f(y) \phi_i(y) dy \right) e^{\lambda_i t} \phi_i(x) \]
Solving the heat equation (3)

Switch sum and integral

\[ u(x, t) = \sum_{i=0}^{\infty} \langle f, \phi_i \rangle e^{\lambda_i t} \phi_i(x) \]

\[ = \sum_{i=0}^{\infty} \left( \int_M f(y) \phi_i(y) dy \right) e^{\lambda_i t} \phi_i(x) \]

\[ = \int_M \left( \sum_{i=0}^{\infty} e^{\lambda_i t} \phi_i(y) \phi_i(x) \right) f(y) dy \]
Solving the heat equation (3)

Compare coefficients

\[ u(x, t) = \sum_{i=0}^{\infty} \langle f, \phi_i(x) \rangle e^{\lambda_i t} \phi_i(x) \]

\[ = \sum_{i=0}^{\infty} \left( \int_{M} f(y) \phi_i(y) dy \right) e^{\lambda_i t} \phi_i(x) \]

\[ = \int_{M} \left( \sum_{i=0}^{\infty} e^{\lambda_i t} \phi_i(y) \phi_i(x) \right) f(y) dy \]

\[ = \int_{M} k_t(x, y) f(y) dy \]
Heat Kernel

\[ k_t(x, y) = \sum_{i=0}^{\infty} e^{\lambda_i t} \phi_i(x) \phi_i(y) \]

We can plot the heat distribution of one specific point \( z \) by using the initial distribution

\[ f(x) = \delta_z \]

\[ u_t(x, t) = \int_M k_t(x, y) \delta_z dy = k_t(x, z) \]
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Heat Kernel Signature

\[ HKS(x, t) = k_t(x, x) = \sum_{i=0}^{\infty} e^{\lambda_i t} (\phi_i(x))^2 \]

- Weighted sum of the squares of eigenfunctions
  - Not sensitive to the order of the eigenfunctions
  - Not sensitive to the sign of the eigenfunctions
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Comparing two Signatures

Discrete Samplings
- Equidistant samplings
- Logarithmic samplings

Distance function
- Using a norm
- Weighted scalar product
Euclidean distance

Equidistant samplings \((t_{i+1} - t_i = \text{const}, \ t_i \in [1; 2.5 \cdot 10^5])\)

\[
d(x, x') = \left( \sum_{i=1}^{N} \left( k_{t_i}(x, x) - k_{t_i}(x', x') \right)^2 \right)^{\frac{1}{2}}
\]
Scaled Heat Kernel Signature

Use logarithmic samples \((\log(t_{i+1}) - \log(t_i) = const)\)

Use the heat trace for weights

\[
\int_M k_t(x, x)dx = \sum_{i=1}^{\infty} e^{\lambda_i \cdot t}
\]

\[
d_{[t_1, t_2]}(x, x') = \sqrt{\int_{t_1}^{t_2} \left( \frac{|k_t(x, x) - k_t(x', x')|}{\int_M k_t(y, y)dy} \right)^2 d \log t}
\]
Scaled Heat Kernel Signature

N=5

N=10
Scaled Heat Kernel Signature
Multi Scale Matching

HKS is based on heat diffusion

For bounded time interval $[t_1; t_2]$ heat distribution is almost the same for full model and partial model.
Multi Scale Error
Conclusion

Heat Kernel Signature has a lot of useful properties

- Good comparison among different shapes
- Multi scale
- Robust under noise
- Stable under isometric deformations

Further Improvements:

Scale Independent Heat Kernel Signature

- Scaling the manifold by $\beta$ result in a amplitude scaling and a time shift by $\beta^2$
Thank you for your Attention

ANY QUESTIONS?