

Exercise Sheet 3

Room: 02.09.023

Tue, 13.05.2014, 13:45-15:15

Submission deadline: Mon, 12.05.2014, 23:59 to windheus@in.tum.de

Mathematics: Differential Geometry

Exercise 1 (One point). *Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth function. We consider its graph*

$$\mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{R}^3, (u, v) \mapsto (u, v, g(u, v))$$

1. *Show that \mathbf{x} is regular and therefore a parametrized surface element.*
2. *Derive the first fundamental form of \mathbf{x} .*
3. *In what cases is the parametrization \mathbf{x} orthogonal (conformal, isometric)?*

Exercise 2 (One point). *Let $g : I \rightarrow \mathbb{R}^+$ be a smooth function. Consider*

$$\mathbf{x} : I \times \mathbb{R} \rightarrow \mathbb{R}^3, (u, v) \mapsto (u, \cos(v)g(u), \sin(v)g(u)).$$

1. *Give an interpretation of \mathbf{x} .*
2. *Show that \mathbf{x} is regular and derive its first fundamental form.*
3. *Calculate the area $A(\mathbf{x}|_{I \times (0, 2\pi)})$.*
4. *Calculate the surface area of a sphere with radius r .*

Exercise 3 (One point). *Find a regular parametrization $\mathbf{x} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ of the torus*

$$T = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \left(\sqrt{x^2 + y^2} - a \right)^2 + z^2 = r^2 \right\}$$

with $a > r > 0$.

Programming: Quadratic Assignment Problem

Exercise 4 (One point). Download and expand the file `exercise3.zip` from the lecture website. Modify the files `distortioncostmatrix.m`, `spectralrelax.m`, and `gametheoretic.m` to implement the functions as explained below. You can run the script `exercise.m` to test and visualize your solutions.

Given two finite sets X, Y , where $|X| = |Y| = n$, equipped with metrics d_X, d_Y , we can represent a correspondence between X, Y as a binary matrix $R \in \{0, 1\}^{n \times n}$, such that $R\mathbf{1} = \mathbf{1}$ and $R^\top \mathbf{1} = \mathbf{1}$. The lecture introduced a relaxation to the Gromov-Hausdorff distortion measure by

$$\text{distortion}^p(R) = \sum_{i,j,k,l} C_{(il)(jk)}^p R_{ij} R_{kl}, \text{ where } C^p \in \mathbb{R}^{n^2 \times n^2} \text{ and} \quad (1)$$

$$C_{(il)(jk)}^p = |d_X(x_i, x_j) - d_Y(y_l, y_k)|^p. \quad (2)$$

This leads to the Quadratic Assignment Problem:

$$\begin{aligned} \min_{R \in \{0,1\}^{n \times n}} \quad & \text{vec}(R)^\top C \text{vec}(R) \\ \text{s.t.} \quad & R\mathbf{1} = \mathbf{1} \\ & R^\top \mathbf{1} = \mathbf{1}. \end{aligned} \quad (3)$$

1. Implement function `distortioncostmatrix` that returns matrix $C^p \in \mathbb{R}^{n^2 \times n^2}$ given metrics d_X, d_Y , and $p \in \mathbb{R}$.

Be careful about the indexing. Since R is reshaped into $\text{vec}(R)$ by stacking the columns of R it holds $\text{vec}(R)_{i+nj} = R_{i,j}$. Analogously $C_{(il)(jk)}^p \hat{=} C_{i+nl, j+nk}^p$.

2. Implement the power iteration scheme introduced in the lecture to compute the optimum of the spectral relaxation:

$$\begin{aligned} \max_{R \in [0,1]^{n \times n}} \quad & \text{vec}(R)^\top \hat{C} \text{vec}(R) \\ \text{s.t.} \quad & \|R\|_2 = 1, \end{aligned} \quad (4)$$

Where $\hat{C} = \max(C) - C$. (Note that the conversion to a maximization problem makes the optimization problem solveable to the power iteration scheme.) Place the implementation into function `spectralrelax`. Function `spectralrelax` has four arguments, `C`, `x0`, `maxI`, and `epsilon`, and returns `R`. The function solves optimization problem (4) specified by `C` and returns the arg min as `R`. `x0` specifies the starting point of the iterations. The iterations should be stopped if $i > \text{maxI}$ or $|R^i - R^{i-1}| < \text{epsilon}$.

3. Implement the iterative scheme introduced in the lecture to compute an (locally optimal) solution of the game-theoretic relaxation:

$$\begin{aligned} \max_{R \in [0,1]^{n \times n}} \quad & \text{vec}(R)^\top \hat{C} \text{vec}(R) \\ \text{s.t.} \quad & \|R\|_1 = 1. \end{aligned} \quad (5)$$

Place the implementation into function `gametheoretic`. Function `gametheoretic` has four arguments, `C`, `x0`, `maxI`, and `epsilon`, and returns `R`. The function solves optimization problem (5) specified by `C` and returns the arg max as `R`. `x0` specifies the starting point of the iterations. The iterations should be stopped if $i > \text{maxI}$ or $|R^i - R^{i-1}| < \text{epsilon}$.